CHAPTER

11 Similarity

Nobody can draw a line that is not a boundary line, every line separates a unity into a multiplicity. In addition, every closed contour no matter what its shape, pure circle or whimsical splash accidental in form, evokes the sensation of "inside" and "outside," followed quickly by the suggestion of "nearby" and "far off," of object and background.

M. C. ESCHER

Path of Life I, M. C. Escher, 1958

OBJECTIVES

In this chapter you will

- review ratio and proportion
- define similar polygons and solids
- discover shortcuts for similar triangles
- learn about area and volume relationships in similar polygons and solids
- use the definition of similarity to solve problems

USING YOUR ALGEBRA SKILLS 9

• USING YOUR ALGEBRA SKILLS 9 • USING Y

Proportion and Reasoning

Working with similar geometric figures involves ratios and proportions. You may be a little rusty with these topics, so let's review.

A **ratio** is an expression that compares two quantities by division. You can write the ratio of quantity a to quantity b in these three ways:

$\frac{a}{b}$ $\qquad$ a to b $\qquad$ $a{:}b$

In this book you will write ratios in fraction form. As with fractions, you can multiply or divide both parts of a ratio by the same number to get an equivalent ratio.

A **proportion** is a statement of equality between two ratios. The equality $\frac{6}{18} = \frac{1}{3}$ is an example of a proportion. Proportions are useful for solving problems involving comparisons.

EXAMPLE A

In a photograph, Dan is 2.5 inches tall and his sister Emma is 1.5 inches tall. Dan's actual height is 70 inches. What is Emma's actual height?

▶ Solution

The ratio of Dan's height to Emma's height is the same in real life as it is in the photo. Let x represent Emma's height and set up a proportion.

Dan's height in photograph

Dan's actual height

$$\frac{2.5}{1.5} = \frac{70}{x}$$

Emma's height in photograph

Emma's actual height

Find Emma's height by solving for x.

$$\frac{2.5}{1.5} = \frac{70}{x} \qquad \text{Original proportion.}$$

$$\frac{2.5}{1.5}x = 70 \qquad \text{Multiply both sides by } x.$$

$$2.5x = 105 \qquad \text{Multiply both sides by 1.5.}$$

$$x = 42 \qquad \text{Divide both sides by 2.5.}$$

Emma is 42 inches tall.

There are other proportions you could have used to solve the problem in Example A. For instance, the ratio of Dan's actual height to his height in the photo is equal to the ratio of Emma's actual height to her height in the photo. So you could have found Emma's height by solving $\frac{70}{2.5} = \frac{x}{1.5}$. What other correct proportion could you use?

Some proportions require more algebra to solve.

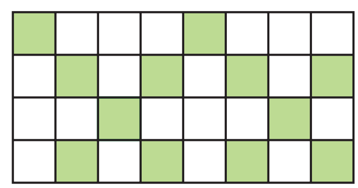
EXAMPLE B Solve $\frac{306}{24} = \frac{x + 50}{20}$.

▶ Solution

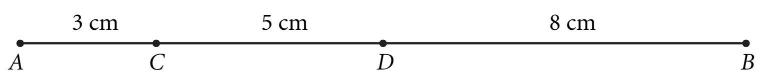
$\frac{306}{24} = \frac{x + 50}{20}$	Original proportion.
$20 \cdot \frac{306}{24} = x + 50$	Multiply both sides by 20.
$255 = x + 50$	Multiply and divide on the left side.
$205 = x$	Subtract 50 from both sides.

EXERCISES

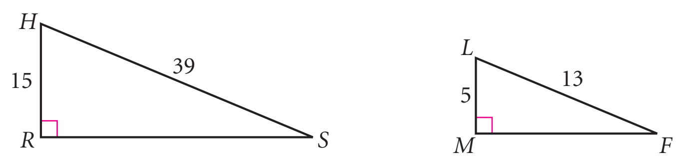
1. Look at the rectangle at right. Find the ratio of the shaded area to the area of the whole figure. Find the ratio of the shaded area to the unshaded area.



2. Use the figure below to find these ratios: $\frac{AC}{CD}$, $\frac{CD}{BD}$, and $\frac{BD}{BC}$.



3. Consider these triangles.



a. Find the ratio of the perimeter of $\triangle RSH$ to the perimeter of $\triangle MFL$.

b. Find the ratio of the area of $\triangle RSH$ to the area of $\triangle MFL$.

In Exercises 4–12, solve the proportion.

4. $\frac{7}{21} = \frac{a}{18}$

5. $\frac{10}{b} = \frac{15}{24}$

6. $\frac{20}{13} = \frac{60}{c}$

7. $\frac{4}{5} = \frac{x}{7}$

8. $\frac{2}{y} = \frac{y}{32}$

9. $\frac{14}{10} = \frac{x + 9}{15}$

10. $\frac{10}{10 + z} = \frac{35}{56}$

11. $\frac{d}{5} = \frac{d + 3}{20}$

12. $\frac{y}{y + 2} = \frac{15}{21}$

13. Solve this proportion for x. Assume $c \neq 0$ and $z \neq 0$.

$$\frac{x}{c} = \frac{b}{z}$$

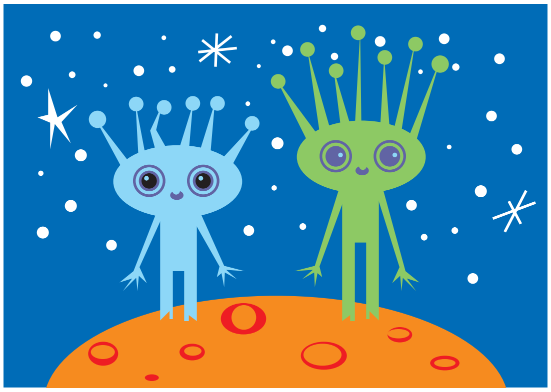
In Exercises 14–17, use a proportion to solve the problem.

14. **APPLICATION** A car travels 106 miles on 4 gallons of gas. How far can it go on a full tank of 12 gallons?

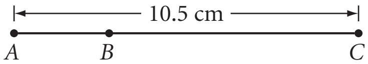
15. **APPLICATION** Ernie is a baseball pitcher. He gave up 34 runs in 152 innings last season. What is Ernie's earned run average—the number of runs he would give up in 9 innings? Give your answer accurate to two decimal places.

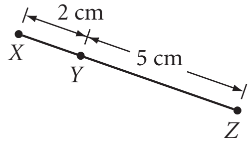
16. **APPLICATION** The floor plan of a house is drawn to the scale of $\frac{1}{4}$ in. = 1 ft. The master bedroom measures 3 in. by $3\frac{3}{4}$ in. on the blueprints. What is the actual size of the room?

17. Altor and Zenor are ambassadors from Titan, the largest moon of Saturn. The sum of the lengths of any Titan's antennae is a direct measure of that Titan's age. Altor has antennae with lengths 8 cm, 10 cm, 13 cm, 16 cm, 14 cm, and 12 cm. Zenor is 130 years old, and her seven antennae have an average length of 17 cm. How old is Altor?



18. Assume $\frac{AB}{XY} = \frac{BC}{YZ}$. Find AB and BC.





IMPROVING YOUR ALGEBRA SKILLS

Algebraic Magic Squares II

In this algebraic magic square, the sum of the entries in every row, column, and diagonal is the same. Find the value of x.

$8 - x$	15	14	$11 - x$
12	$x - 1$	x	9
8	$x + 3$	$x + 4$	5
$2x - 1$	3	2	$2x + 2$

LESSON

11.1

Similar Polygons

He that lets
the small things bind him
Leaves the great
undone behind him.

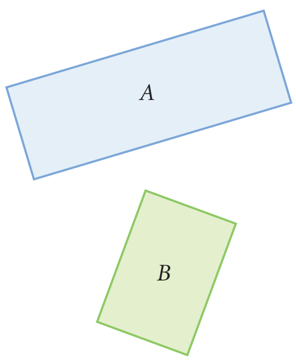
PIET HEIN

You know that figures that have the same shape and size are congruent figures. Figures that have the same shape but not necessarily the same size are **similar figures.** To say that two figures have the same shape but not necessarily the same size is not, however, a precise definition of similarity.

Is your reflection in a fun-house mirror similar to a regular photograph of you? The images have a lot of features in common, but they are not mathematically similar. In mathematics, you can think of similar shapes as enlargements or reductions of each other with no irregular distortions.

Are all rectangles similar? They have common characteristics, but they are not all similar. That is, you could not enlarge or reduce a given rectangle to fit perfectly over every other rectangle. What about other geometric figures: squares, circles, triangles?

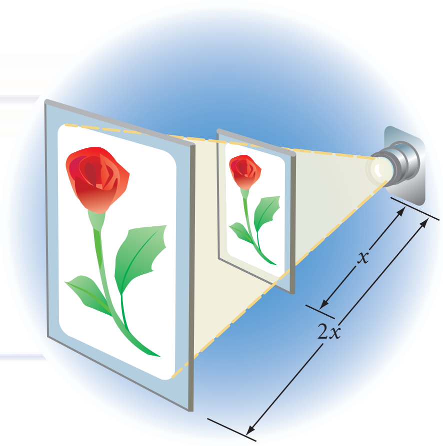
The uneven surface of a fun-house mirror creates a distorted image of you. Your true proportions look different in your reflection.



Rectangles *A* and *B* are not similar. You could not enlarge or reduce one to fit perfectly over the other.

Art CONNECTION

Movie scenes are scaled down to small images on strips of film. Then they are scaled up to fit a large screen. So the film image and the projected image are similar. If the distance between the projector and the screen is decreased by half, each dimension of the screen image is cut in half.

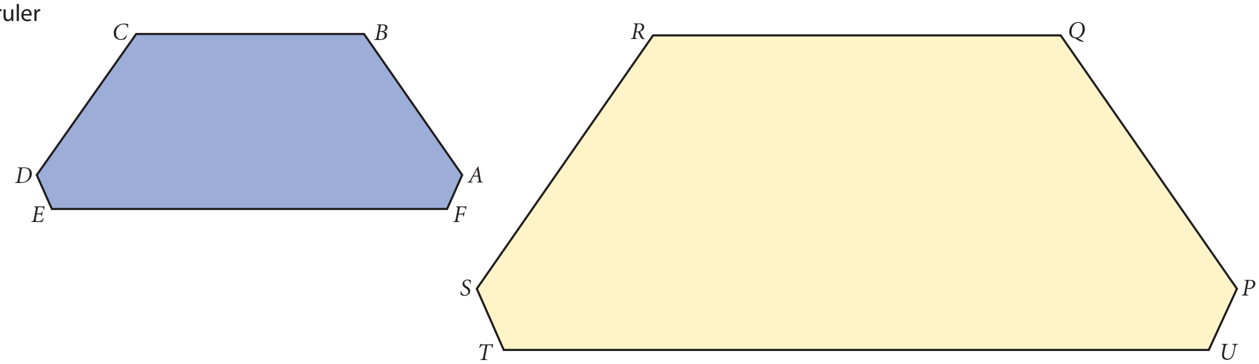


Investigation 1
What Makes Polygons Similar?

You will need

- patty paper
- a ruler

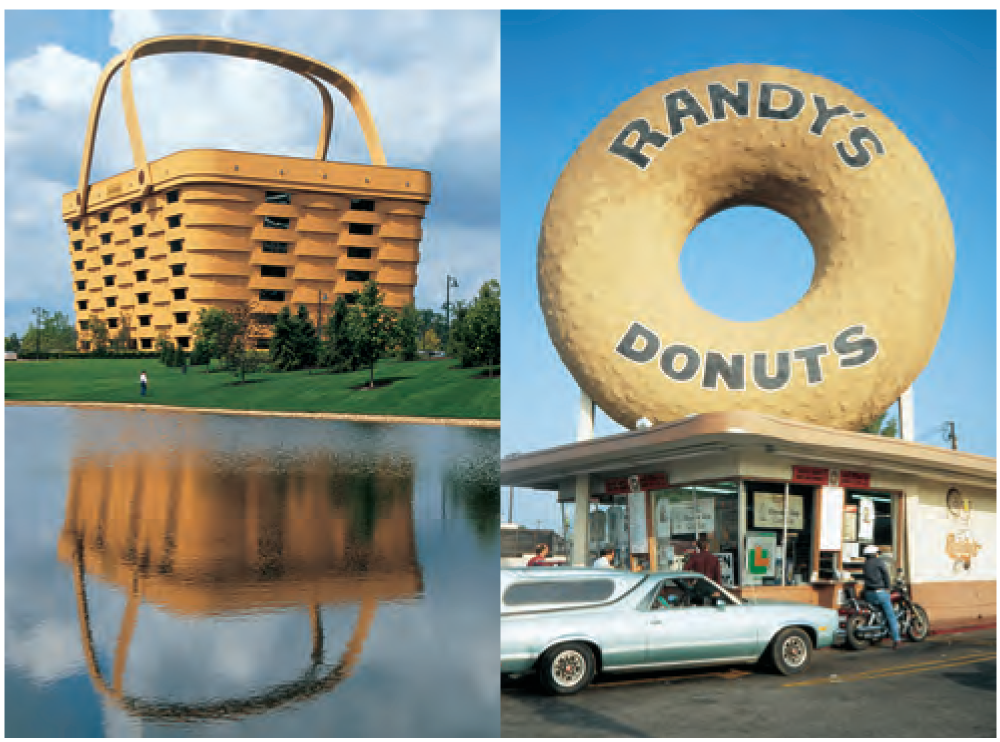
Let's explore what makes polygons similar. Hexagon *PQRSTU* is an enlargement of hexagon *ABCDEF*—they are similar.



Step 1 Use patty paper to compare all corresponding angles. How do the corresponding angles compare?

Step 2 Measure the corresponding segments in both hexagons.

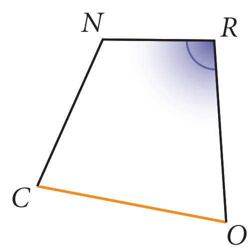
Step 3 Find the ratios of the lengths of corresponding sides. How do the ratios of corresponding sides compare?



Similar objects are often used to create unique buildings. The giant basket shown here is actually an office building for a basket manufacturer. The giant donut advertises a donut shop in Los Angeles, California.

From the investigation, you should be able to state a mathematical definition of similar polygons. Two polygons are **similar polygons** if and only if the corresponding angles are congruent and the corresponding sides are proportional. Similarity is the state of being similar.

The statement $CORN \sim PEAS$ says that quadrilateral $CORN$ is similar to quadrilateral $PEAS$. Just as in statements of congruence, the order of the letters tells you which segments and which angles in the two polygons correspond.



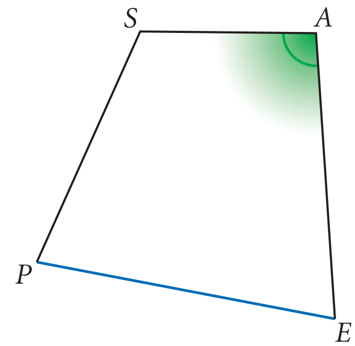
Corresponding angles are congruent:

$\angle C \cong \angle P \qquad \angle R \cong \angle A$

$\angle O \cong \angle E \qquad \angle N \cong \angle S$

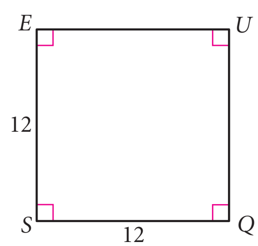
Corresponding segments are proportional:

$$\frac{CO}{PE} = \frac{OR}{EA} = \frac{RN}{AS} = \frac{NC}{SP}$$

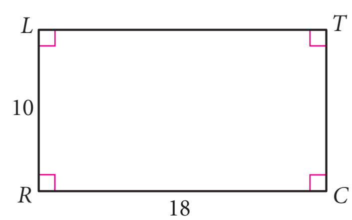


Do you need both conditions—congruent angles and proportional sides—to guarantee that the two polygons are similar? For example, if you know only that the corresponding angles of two polygons are congruent, can you conclude that the polygons have to be similar? Or, if corresponding sides of two polygons are proportional, are the polygons necessarily similar? These counterexamples show that both answers are no.

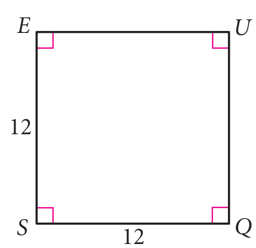
In the figures below, corresponding angles of square $SQUE$ and rectangle $RCTL$ are congruent, but their corresponding sides are not proportional.



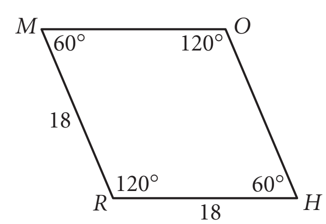
$$\frac{12}{10} \neq \frac{12}{18}$$



In the figures below, corresponding sides of square $SQUE$ and rhombus $RHOM$ are proportional, but their corresponding angles are not congruent.



$$\frac{12}{18} = \frac{12}{18}$$

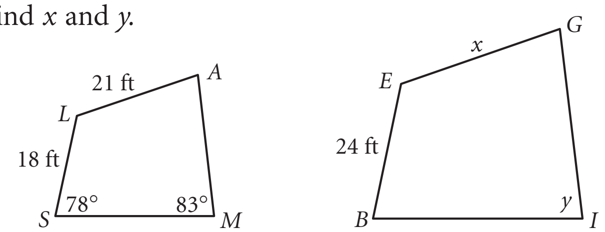


Clearly, neither pair of polygons is similar. You cannot conclude that two polygons are similar given only the fact that their corresponding angles are congruent or given only the fact that their corresponding sides are proportional.

You can use the definition of similar polygons to find missing measures in similar polygons.

EXAMPLE

SMAL ~ *BIGE*
Find x and y.



▶ Solution

The quadrilaterals are similar, so you can use a proportion to find x.

$\frac{18}{24} = \frac{21}{x}$	A proportion of corresponding sides.
$18x = (24)(21)$	Multiply both sides by $24x$ and reduce.
$x = 28$	Divide both sides by 18.

The measure of the side labeled x is 28 ft.

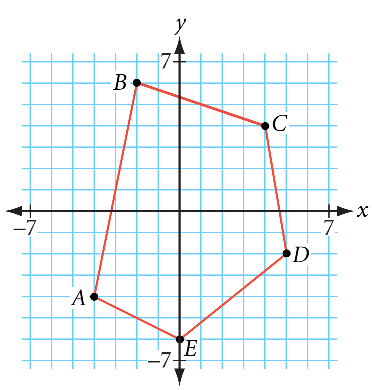
In similar polygons, corresponding angles are congruent, so $\angle M \cong \angle I$. The measure of the angle labeled y is therefore 83°.

Earlier in this book you worked with translations, rotations, and reflections. These rigid transformations preserve both size and shape—the images are congruent to the original figures. One type of nonrigid transformation is called a **dilation.** Let's look at an image after a dilation transformation.

Investigation 2
Dilations on a Coordinate Plane

You will need

- graph paper
- a straightedge
- patty paper
- a compass



Step 1 To dilate a pentagon on a coordinate plane, first copy this pentagon onto your graph paper.

Step 2 Have each member of your group multiply the coordinates of the vertices by one of these numbers: $\frac{1}{2}$, $\frac{3}{4}$, 2, or 3. Each of these factors is called a **scale factor.**

Step 3 Locate these new coordinates on your graph paper and draw the new pentagon.

Step 4 Copy the original pentagon onto patty paper. Compare the corresponding angles of the two pentagons. What do you notice?

Step 5 Compare the corresponding sides with a compass or with patty paper. The length of each side of the new pentagon is how many times as long as the length of the corresponding side of the original pentagon?

Step 6 Compare results with your group. You should be ready to state a conjecture.

Dilation Similarity Conjecture

C-92

If one polygon is the image of another polygon under a dilation, then $\underline{\ ?\ }$.

History CONNECTION

Similarity plays an important role in human history. For example, accurate maps of regions of China have been found dating back to the second century B.C.E. Neolithic cave paintings 8000–6000 B.C.E. contain small-scale drawings of the animals people hunted. Giant geoglyphs made by the Nazca people of Peru (110 B.C.E.–800 C.E.) are some of the largest scale drawings ever made.

In order for a map to be accurate, cartographers need to use similarity to reduce the earth's attributes to a smaller scale. This sixteenth-century French map, a plan of Constantinople, included a mariner's chart of America, Europe, Africa, and Asia.

This cave art is part of a grouping of over 15,000 drawings in Tassili N'Ajjier National Park of the Algerian Sahara. Interestingly, these drawings depict animals and landscapes that are absent from the region today, such as these elephants or vast lakes.

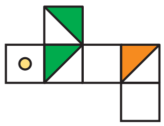
This monkey is a geoglyph found in the Pampa region of Peru in 1920. Called the Nazca Lines, the figure measures over 400 feet long and can only be clearly seen from the air.

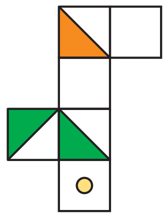
EXERCISES

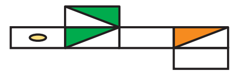
You will need

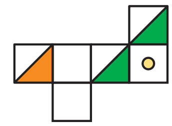
Construction tools for Exercises 22 and 25

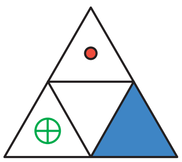
For Exercises 1 and 2, match the similar figures.

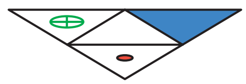
1. 

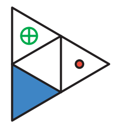
A. 

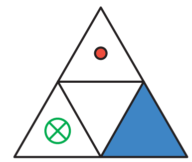
B. 

C. 

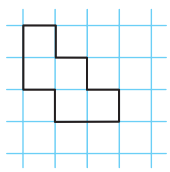
2. 

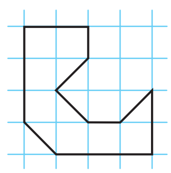
A. 

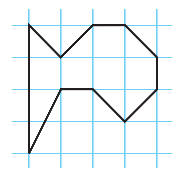
B. 

C. 

For Exercises 3–5, sketch on graph paper a similar, but not congruent, figure.

3. 

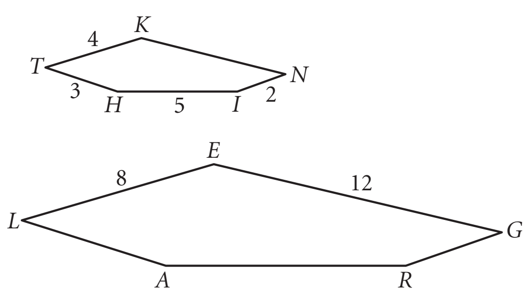
4. 

5. 

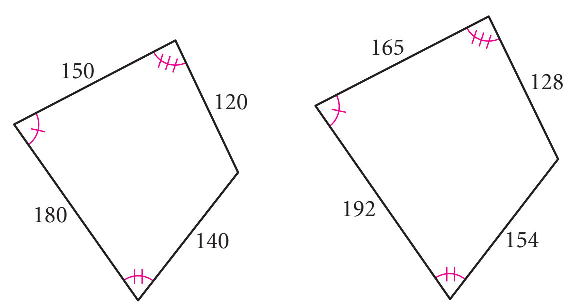
6. Complete the statement: If Figure A is similar to Figure B and Figure B is similar to Figure C, then __?__. Draw and label figures to illustrate the statement.

For Exercises 7–14, use the definition of similar polygons. All measurements are in centimeters.

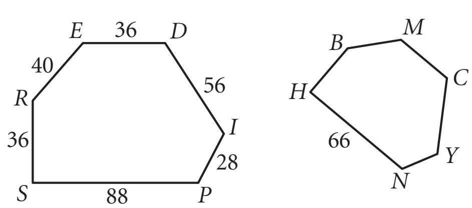
7. *THINK* ~ *LARGE*
Find *AL*, *RA*, *RG*, and *KN*.



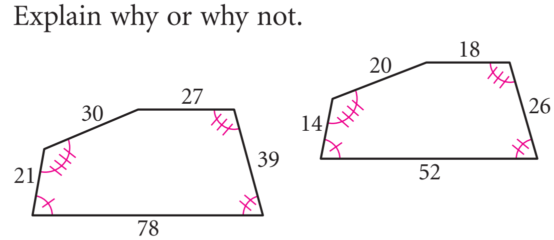
8. Are these polygons similar?
Explain why or why not. ⓗ



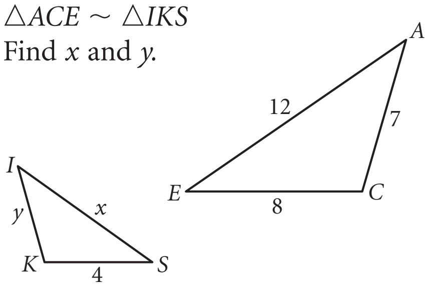
9. *SPIDER* ~ *HNYCMB*
Find *NY*, *YC*, *CM*, and *MB*.



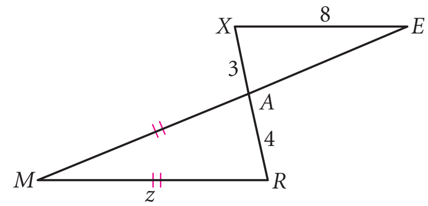
10. Are these polygons similar?
Explain why or why not.



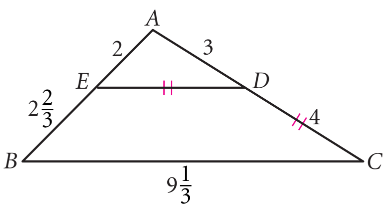
11. $\triangle ACE \sim \triangle IKS$
Find x and y.



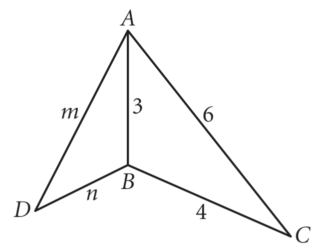
12. $\triangle RAM \sim \triangle XAE$
Find z.



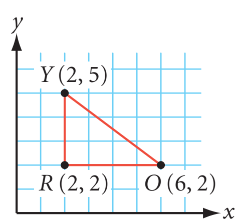
13. $\overline{DE} \parallel \overline{BC}$
Are the corresponding angles congruent in $\triangle AED$ and $\triangle ABC$? Are the corresponding sides proportional? Is $\triangle AED \sim \triangle ABC$? ⓗ



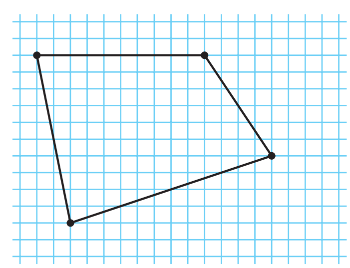
14. $\triangle ABC \sim \triangle DBA$
Find m and n.



15. Copy $\triangle ROY$ onto your graph paper. Sketch its dilation with a scale factor of 3. What is the ratio of the perimeter of the smaller triangle to the perimeter of the larger triangle?



16. Copy this quadrilateral onto your graph paper. Draw a similar quadrilateral with each side half the length of its corresponding side in the original quadrilateral.



17. **APPLICATION** The photo at right shows the Crazy Horse Memorial and a scale model of the complete monument's design. The head of the Crazy Horse Memorial, from the chin to the top of the forehead, is 87.5 ft high. When the arms are carved, how long will each be? Use the photo and explain how you got your answer.

The Crazy Horse Memorial is located in South Dakota. Started in 1948, it will be the world's largest sculpture when complete. You can learn more about this monument using the links at www.keymath.com/DG .

Review

For Exercises 18–20, use algebra to answer each proportion question. Assume that a, b, c, and d are all nonzero values.

18. If $\frac{15}{a} = \frac{20}{a + 12}$, then $a = \underline{\ ?\ }$.

19. If $\frac{a}{b} = \frac{c}{d}$, then $ad = \underline{\ ?\ }$.

20. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \underline{\ ?\ }$.

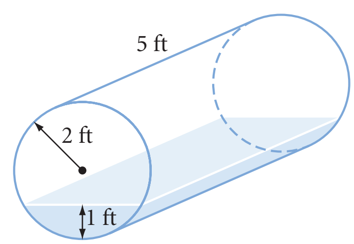
21. **APPLICATION** Jade and Omar each put in \$1000 to buy an old boat to fix up. Later Jade spent \$825 on materials, and Omar spent \$1650 for parts. They worked an equal number of hours on the boat and eventually sold it for \$6800. How might they divide the \$6800 fairly? Explain your reasoning.

22. *Construction* Use a compass and straightedge to construct

a. A rhombus with a 60° angle.

b. A second rhombus of different size with a 60° angle.

23. A cubic foot of liquid is about 7.5 gallons. How many gallons of liquid are in this tank?

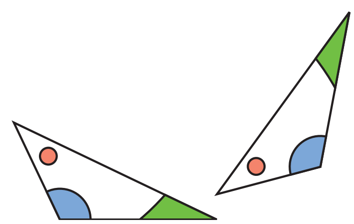


24. Triangle PQR has side lengths 18 cm, 24 cm, and 30 cm. Is $\triangle PQR$ a right triangle? Explain why or why not.

25. *Construction* Use the triangular figure at right and its rotated image. ⓗ

a. Copy the figure and its image onto a piece of patty paper. Locate the center of rotation. Explain your method.

b. Copy the figure and its image onto a sheet of paper. Locate the center of rotation using a compass and straightedge. Explain your method.



Career CONNECTION

Similarity plays an important part in the design of cars, trucks, and airplanes, which is done with small-scale drawings and models.

This model airplane is about to be tested in a wind tunnel.

project

MAKING A MURAL

A mural is a large work of art that usually fills an entire wall. This project gives you a chance to use similarity and make your own mural.

One way to create a mural from a small picture is to draw a grid of squares lightly over the small picture. Then divide the mural surface into a similar but larger grid. Proceeding square by square, draw the lines and curves of the small grid in each corresponding square of the mural grid. Complete the mural by coloring or painting the regions and erasing the grid lines.

You project should include

- An original drawing, a cartoon, or a photograph divided into a grid of squares.
- A finished mural drawn on a large sheet of paper.

You can learn more about the art of mural making through the links at www.keymath.com/DG .

Mural artists use similarity to help them create large artwork. This mural, finished in 1990, is in the North Beach neighborhood of San Francisco, California.

LESSON

11.2

Similar Triangles

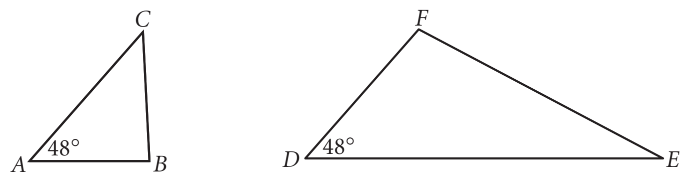
Life is change. Growth is optional. Choose wisely.

KAREN KAISER CLARK

In Lesson 11.1, you concluded that you must know about both the angles and the sides of two quadrilaterals in order to make a valid conclusion about their similarity.

However, triangles are unique. Recall from Chapter 4 that you found four shortcuts for triangle congruence: SSS, SAS, ASA, and SAA. Are there shortcuts for triangle similarity as well? Let's first look for shortcuts using only angles.

The figures below illustrate that you cannot conclude that two triangles are similar given that only one set of corresponding angles are congruent.



$\angle A \cong \angle D$, but $\triangle ABC$ is not similar to $\triangle DEF$.

How about two sets of congruent angles?

Investigation 1

Is AA a Similarity Shortcut?

You will need

- a compass
- a ruler

If two angles of one triangle are congruent to two angles of another triangle, must the two triangles be similar?

Step 1 Draw any triangle ABC.

Step 2 Construct a second triangle, DEF, with $\angle D \cong \angle A$ and $\angle E \cong \angle B$. What will be true about $\angle C$ and $\angle F$? Why?

Step 3 Carefully measure the lengths of the sides of both triangles. Compare the ratios of the corresponding sides. Is $\frac{AB}{DE} \approx \frac{AC}{DF} \approx \frac{BC}{EF}$?

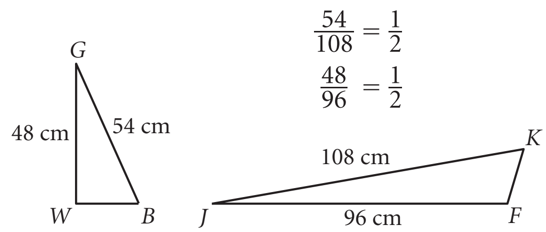
Step 4 Compare your results with the results of others near you. You should be ready to state a conjecture.

AA Similarity Conjecture C-93

If $\underline{\ ?\ }$ angles of one triangle are congruent to $\underline{\ ?\ }$ angles of another triangle, then $\underline{\ ?\ }$.

As you may have guessed from Step 2 of the investigation, there is no need to investigate the AAA Similarity Conjecture. Thanks to the Third Angle Conjecture, the AA Similarity Conjecture is all you need.

Now let's look for shortcuts for similarity that use only sides. The figures below illustrate that you cannot conclude that two triangles are similar given that two sets of corresponding sides are proportional.



$\frac{GB}{JK} = \frac{GW}{JF}$, but $\triangle GWB$ is not similar to $\triangle JFK$.

How about all three sets of corresponding sides?

Investigation 2
Is SSS a Similarity Shortcut?

You will need
- a compass
- a straightedge
- a protractor

If three sides of one triangle are proportional to the three sides of another triangle, must the two triangles be similar?

Draw any triangle *ABC*. Then construct a second triangle, *DEF*, whose side lengths are a multiple of the original triangle. (Your second triangle can be larger or smaller.)

Compare the corresponding angles of the two triangles. Compare your results with the results of others near you and state a conjecture.

SSS Similarity Conjecture
C-94

If the three sides of one triangle are proportional to the three sides of another triangle, then the two triangles are $\underline{\ ?\ }$.

Many dollhouses and other toys are scale models of real objects.

In Investigations 1 and 2, you discovered two shortcuts for triangle similarity: AA and SSS. But if AA is a shortcut, then so are ASA, SAA, and AAA. That leaves SAS and SSA as possible shortcuts to consider.

Investigation 3
Is SAS a Similarity Shortcut?

You will need
- a compass
- a protractor
- a ruler

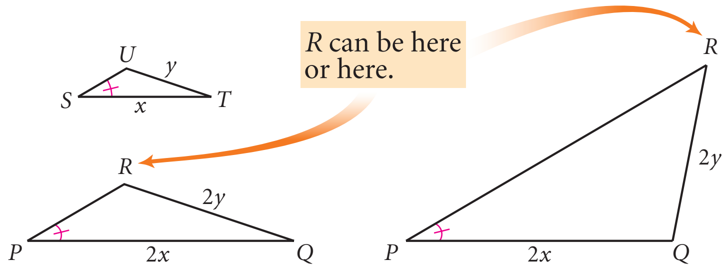
Is SAS a shortcut for similarity? Try to construct two different triangles that are not similar but have two pairs of sides proportional and the pair of included angles equal in measure.

Compare the measures of corresponding sides and corresponding angles. Share your results with others near you and state a conjecture.

SAS Similarity Conjecture
C-95

If two sides of one triangle are proportional to two sides of another triangle and _?_, then the _?_.

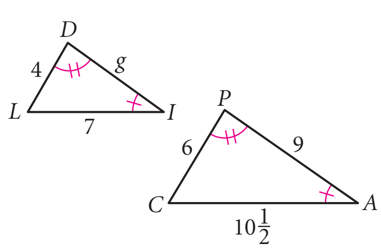
One question remains: Is SSA a shortcut for similarity? Recall from Chapter 4 that SSA did not work for congruence because you could create two different triangles. Those two different triangles were neither congruent nor similar. So, no, SSA is not a shortcut for similarity.



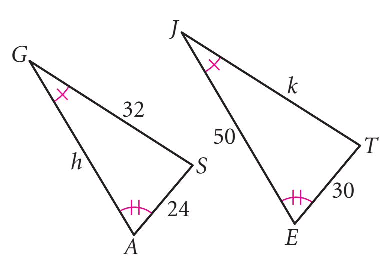
Exercises

For Exercises 1–14, use your new conjectures. All measurements are in centimeters.

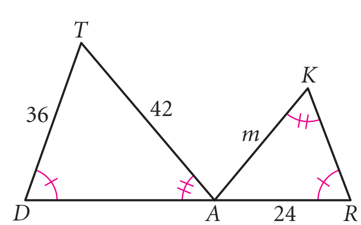
1. $g = \underline{?}$



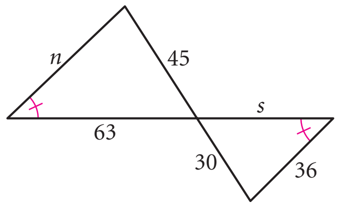
2. $h = \underline{?}$, $k = \underline{?}$



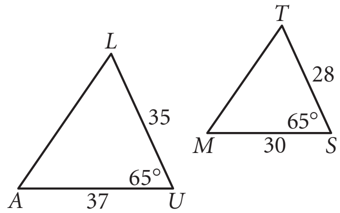
3. $m = \underline{?}$ ⓗ



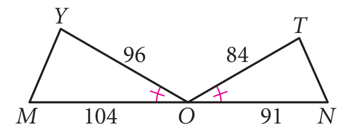
4. $n =$ __?__, $s =$ __?__



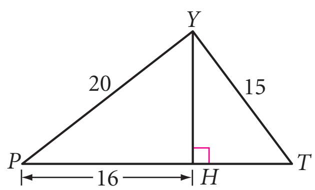
5. Is $\triangle AUL \sim \triangle MST$? Explain why or why not.



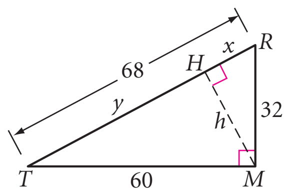
6. Is $\triangle MOY \sim \triangle NOT$? Explain why or why not.



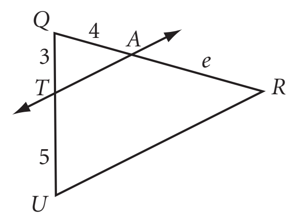
7. Is $\triangle PHY \sim \triangle YHT$? Is $\triangle PTY$ a right triangle? Explain why or why not.



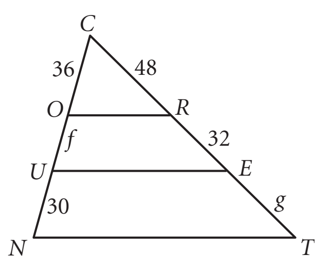
8. Why is $\triangle TMR \sim \triangle THM \sim \triangle MHR$? Find x, y, and h. ⓗ



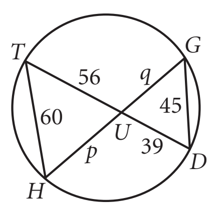
9. $\overline{TA} \parallel \overline{UR}$
Is $\angle QTA \cong \angle TUR$?
Is $\angle QAT \cong \angle ARU$?
Why is $\triangle QTA \sim \triangle QUR$?
$e =$ __?__



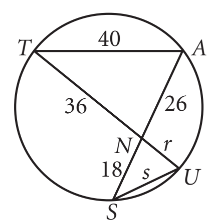
10. $\overline{OR} \parallel \overline{UE} \parallel \overline{NT}$
$f =$ __?__, $g =$ __?__



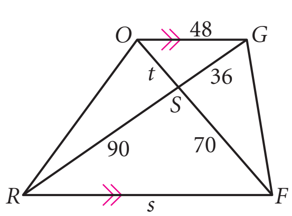
11. Is $\angle THU \cong \angle GDU$?
Is $\angle HTU \cong \angle DGU$?
$p =$ __?__, $q =$ __?__ ⓗ



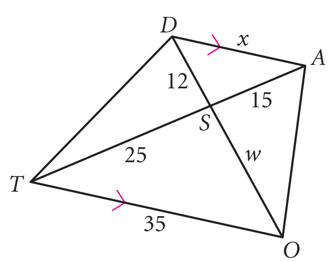
12. Why is $\triangle SUN \sim \triangle TAN$?
$r =$ __?__, $s =$ __?__



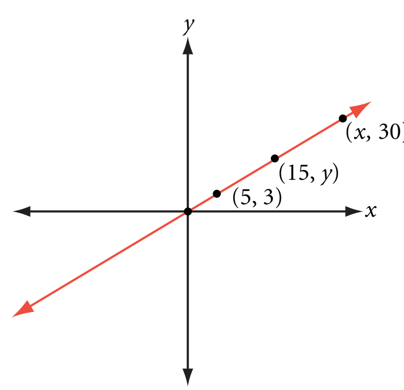
13. *FROG* is a trapezoid.
Is $\angle RGO \cong \angle FRG$?
Is $\angle GOF \cong \angle RFO$?
Why is $\triangle GOS \sim \triangle RFS$?
$t =$ __?__, $s =$ __?__



14. *TOAD* is a trapezoid.
$w =$ __?__, $x =$ __?__

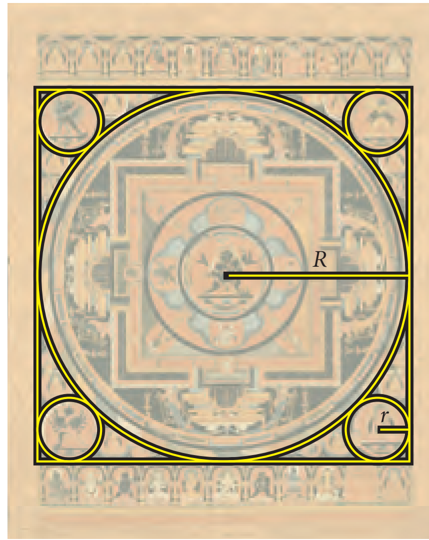


15. Find x and y. ⓗ



Review

16. In the figure below right, find the radius, r, of one of the small circles in terms of the radius, R, of the large circle. ⓗ



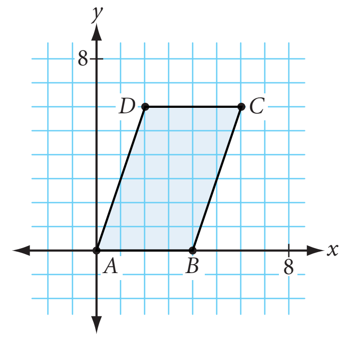
This Tibetan mandala is a complex design with a square inscribed within a circle and tangent circles inscribed within the corners of a larger circumscribed square.

17. **APPLICATION** Phoung volunteers at an SPCA that always houses 8 dogs. She notices that she uses seven 35-pound bags of dry dog food every two months. A new, larger SPCA facility that houses 20 dogs will open soon. Help Phoung estimate the amount of dry dog food that the facility should order every three months. Explain your reasoning.

18. **APPLICATION** Ramon and Sabina are oceanography students studying the habitat of a Hawaiian fish called Humuhumunukunukuapuaʻa. They are going to use the capture-recapture method to determine the fish population. They first capture and tag 84 fish, which they release back into the ocean. After one week, Ramon and Sabina catch another 64. Only 12 have tags. Can you estimate the population of Humuhumunukunukuapuaʻa?

19. Points $A(-9, 5)$, $B(4, 13)$, and $C(1, -7)$ are connected to form a triangle. Find the area of $\triangle ABC$.

20. Use the ordered pair rule, $(x, y) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$, to relocate the coordinates of the vertices of parallelogram $ABCD$. Call the new parallelogram $A'B'C'D'$. Is $A'B'C'D'$ similar to $ABCD$? If they are similar, what is the ratio of the perimeter of $ABCD$ to the perimeter of $A'B'C'D'$? What is the ratio of their areas?



21. The photo below shows a fragment from an ancient statue of the Roman Emperor Constantine. Use this photo to estimate how tall the entire statue was. List the measurements you need to make. List any assumptions you need to make. Explain your reasoning.

History CONNECTION

The Emperor Constantine the Great (Roman Emperor 306–337 C.E.) adopted Christianity as the official religion of the Roman Empire. The Roman Catholic Church regards him as Saint Constantine, and the city of Constantinople was named for him. The colossal statue of Constantine was built between 315 and 330 C.E., and broke when sculptors tried to add the extra weight of a beard to its face. The pieces of the statue remain close to its original location in Rome, Italy.

IMPROVING YOUR VISUAL THINKING SKILLS

Build a Two-Piece Puzzle

Construct two copies of Figure A, shown at right. Here's how to construct the figure.

- Construct a regular hexagon.
- Construct an equilateral triangle on two alternating edges, as shown.
- Construct a square on the edge between the two equilateral triangles, as shown.

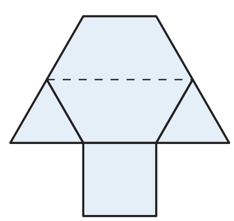


Figure A

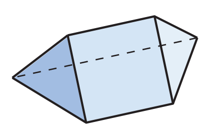


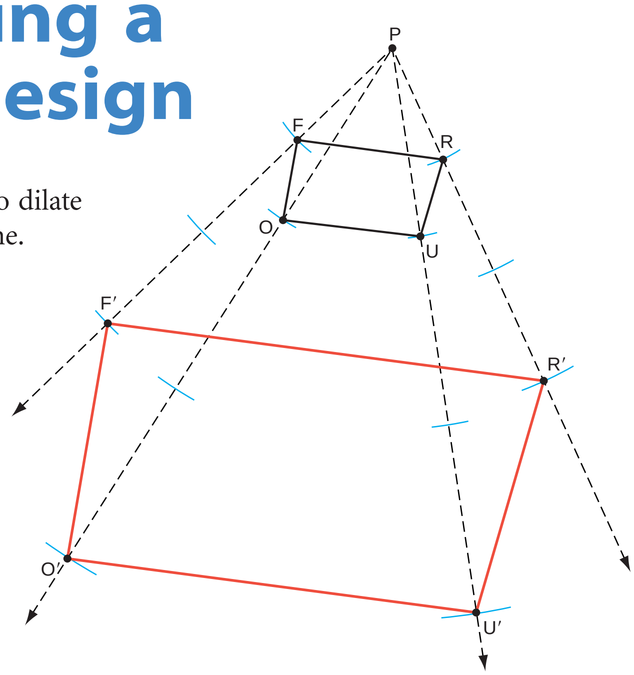
Figure B

Cut out each copy and fold them into two identical solids, as shown in Figure B. Tape the edges. Now arrange your two solids to form a regular tetrahedron.

THE GEOMETER'S SKETCHPAD

Exploration

Constructing a Dilation Design



In Lesson 11.1, you saw how to dilate a polygon on a coordinate plane. You can also use a simple construction to dilate any polygon. Draw rays from any point P through the vertices of the polygon. Use a compass to measure the distance from point P to one of the vertices. Then mark this distance two more times along the ray; that will give you a scale factor of 3. Repeat this process for each of the other vertices using the same scale factor. When you connect the image of each vertex, you will get a similar polygon. Try it yourself. How would you create a similar polygon with a scale factor of 2? Of $\frac{1}{2}$?

Now take a closer look at *Path of Life I,* the M. C. Escher woodcut that begins this chapter. Notice that dilations transform the black fishlike creatures, shrinking them again and again as they approach the picture's center. The same is true for the white fish. (The black-and-white fish around the outside border are congruent to one another, but they're not similar to the other fish.) Also notice that rotations repeat the dilations in eight sectors. With Sketchpad, you can make a similar design.

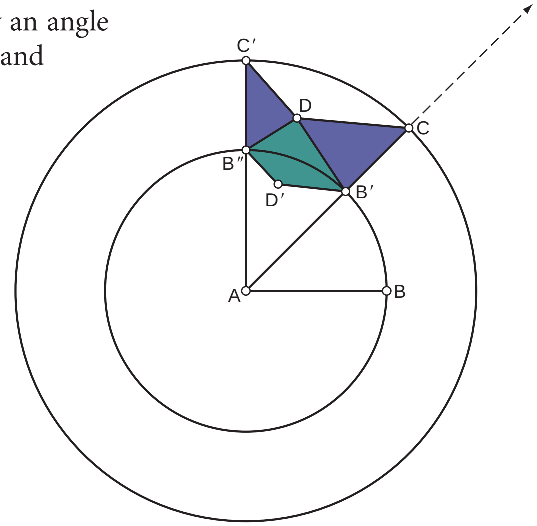
Activity
Dilation Creations

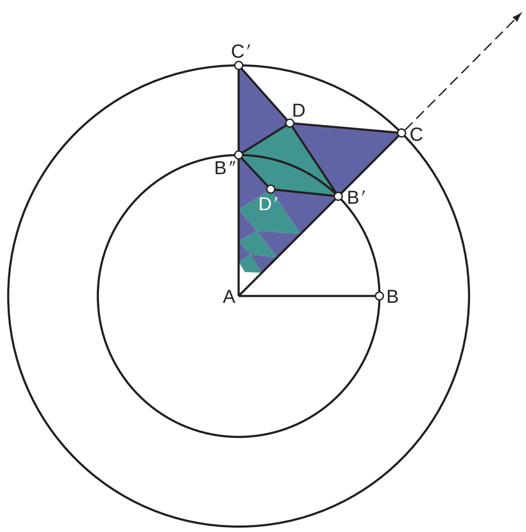
Step 1 Construct a circle with center point A and point B on the circle. Construct $\overline{AB}$.

Step 2 Use the Transform menu to mark point A as center, then rotate point B by an angle of 45°. Your new point is B'. Construct $\overrightarrow{AB'}$.

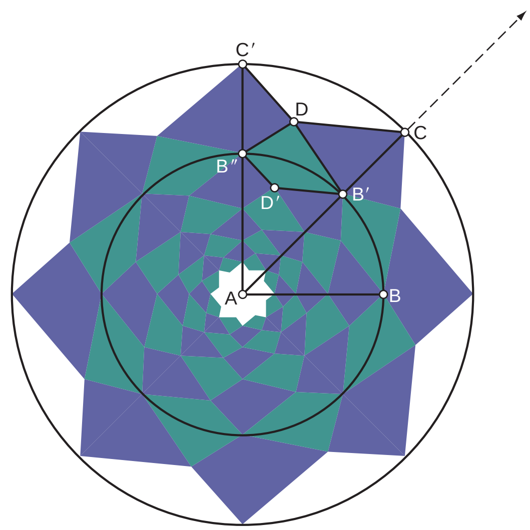
Step 3 Construct a larger circle with center point A and point C on $\overrightarrow{AB'}$. Hide $\overrightarrow{AB'}$ and construct $\overline{AC}$.

Step 4	Rotate $\overline{AC}$, point B', and point C by an angle of 45°. You now have $\overline{AC'}$, point B'', and point C'.
Step 5	Construct $\overline{C'D}$ and $\overline{DC}$, where D is any point in the region between the circles and between $\overline{B''C'}$ and $\overline{B'C}$.
Step 6	Select, in order, $\overline{AB}$ and $\overline{AC}$. Choose **Mark Segment Ratio** from the Transform menu. This marks a ratio of a shorter segment to a longer segment. Because this ratio is less than 1, dilating by this scale factor will shrink objects.
Step 7	Select $\overline{C'D}$, $\overline{DC}$, and point D. Choose **Dilate** from the Transform menu and dilate by the marked ratio. The dilated images are $\overline{B''D'}$, $\overline{D'B'}$, and D'.
Step 8	Construct three polygon interiors—two triangles and a quadrilateral.
Step 9	Select the three polygon interiors and dilate them by the marked ratio. Repeat this process two or three times.





Step 9



Step 10

Step 10	Select all the polygon interiors in the sector and rotate them by an angle of 45°. Repeat the rotation by 45° until you've gone all the way around the circle.

Now you have a design that has the same basic mathematical properties as Escher's *Path of Life I.*

Step 11 Experiment with changing the design by moving different points. Answer these questions.

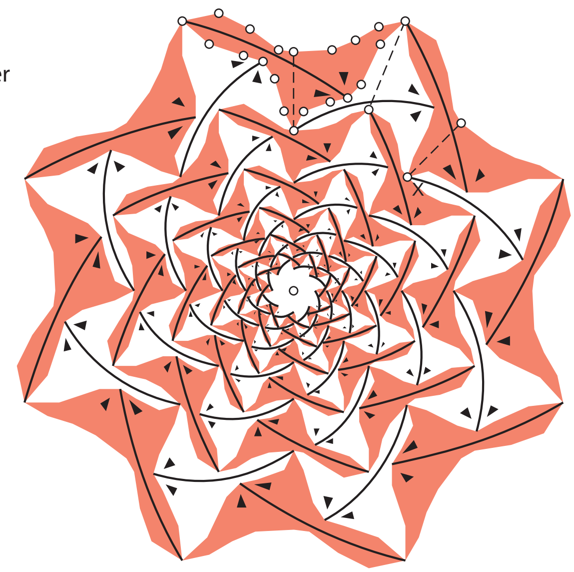
a. What locations of point D result in both rotational and reflectional symmetry?

b. Drag point C away from point A. What does this do to the numerical dilation ratio? What effect does that have on the geometric figure?

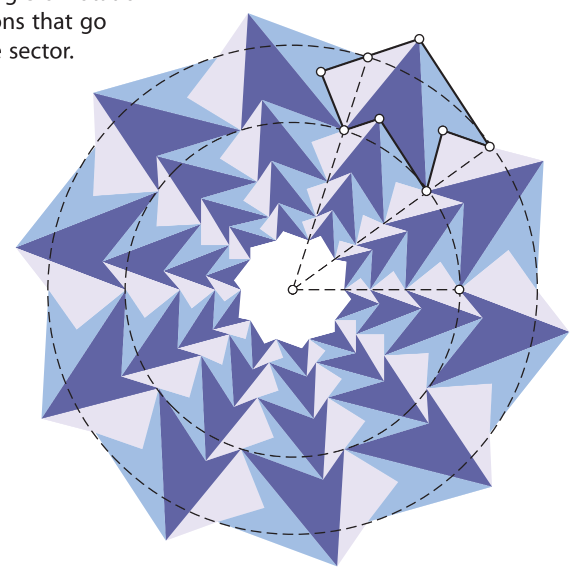
c. Drag point C toward point A. What happens when the circle defined by point C becomes smaller than the circle defined by point B? Why?

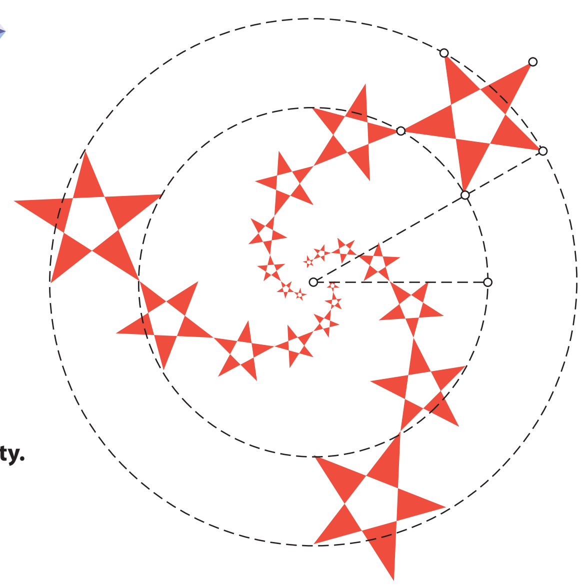
Experiment with other dilation-rotation designs of your own. Try different angles of rotation or different polygons. Here are some examples.

Mathematician Doris Schattschneider of Moravian College is an expert on M. C. Escher and the mathematics he explored. She made this sketch based on *Path of Life I.*



This design uses a different angle of rotation and polygons that go outside the sector.





This design uses a two-step transformation, a dilation followed by a rotation, called a **spiral similarity.**

LESSON

11.3

Indirect Measurement with Similar Triangles

Never be afraid to sit awhile and think.

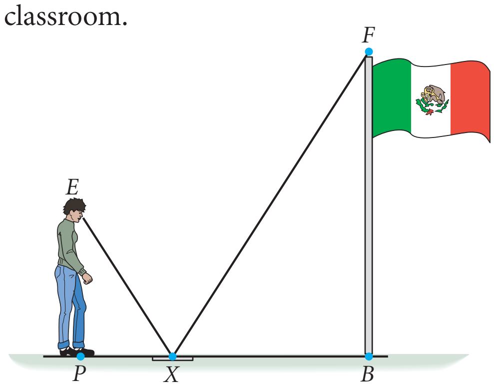
LORRAINE HANSBERRY

You can use similar triangles to calculate the height of tall objects that you can't reach. This is called **indirect measurement.** One method uses mirrors. Try it in the next investigation.

Investigation
Mirror, Mirror

You will need
- metersticks
- masking tape or a soluble pen
- a mirror

Choose a tall object with a height that would be difficult to measure directly, such as a football goalpost, a basketball hoop, a flagpole, or the height of your classroom.



Step 1 Mark crosshairs on your mirror. Use tape or a soluble pen. Call the intersection point X. Place the mirror on the ground several meters from your object.

Step 2 An observer should move to a point P in line with the object and the mirror in order to see the reflection of an identifiable point F at the top of the object at point X on the mirror. Make a sketch of your setup, like this one.

Step 3 Measure the distance PX and the distance from X to a point B at the base of the object directly below F. Measure the distance from P to the observer's eye level, E.

Step 4 Think of $\overline{FX}$ as a light ray that bounces back to the observer's eye along $\overline{XE}$. Why is $\angle B \cong \angle P$? Name two similar triangles. Tell why they are similar.

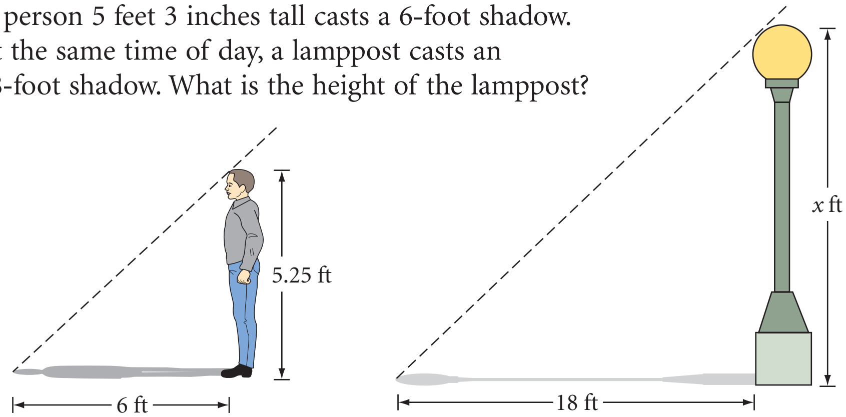
Step 5 Set up a proportion using corresponding sides of similar triangles. Use it to calculate FB, the approximate height of the tall object.

Step 6 Write a summary of what you and your group did in this investigation. Discuss possible causes for error.

Another method of indirect measurement uses shadows.

EXAMPLE

A person 5 feet 3 inches tall casts a 6-foot shadow. At the same time of day, a lamppost casts an 18-foot shadow. What is the height of the lamppost?



▸ Solution

The light rays that create the shadows hit the ground at congruent angles. Assuming both the person and the lamppost are perpendicular to the ground, you have similar triangles by the AA Similarity Conjecture. Solve a proportion that relates corresponding lengths.

$$\frac{5.25}{6} = \frac{x}{18}$$

$$18 \cdot \frac{5.25}{6} = x$$

$$15.75 = x$$

The height of the lamppost is 15 feet 9 inches.

EXERCISES

You will need

Construction tools for Exercise **14**

Geometry software for Exercises **17** and **18**

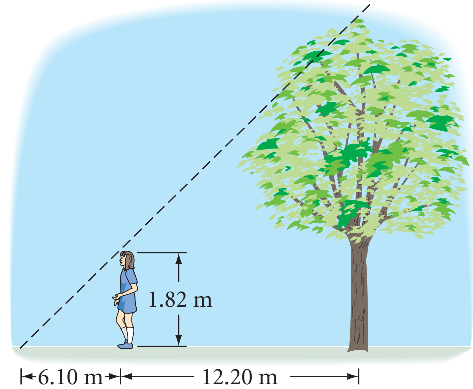
1. A flagpole 4 meters tall casts a 6-meter shadow. At the same time of day, a nearby building casts a 24-meter shadow. How tall is the building?

2. Five-foot-tall Melody casts an 84-inch shadow. How tall is her friend if, at the same time of day, his shadow is 1 foot shorter than hers?

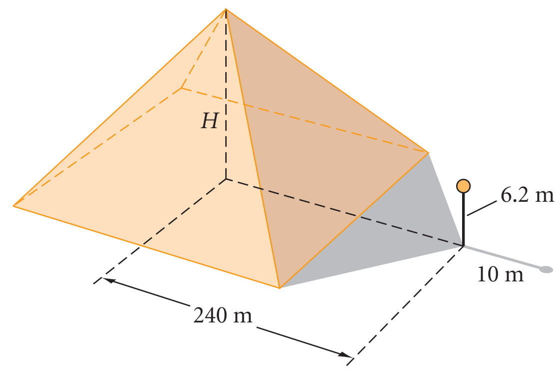
3. A 10 m rope from the top of a flagpole reaches to the end of the flagpole's 6 m shadow. How tall is the nearby football goalpost if, at the same moment, it has a shadow of 4 m? ⓗ

4. Private eye Samantha Diamond places a mirror on the ground between herself and an apartment building and stands so that when she looks into the mirror, she sees into a window. The mirror's crosshairs are 1.22 meters from her feet and 7.32 meters from the base of the building. Sam's eye is 1.82 meters above the ground. How high is the window?

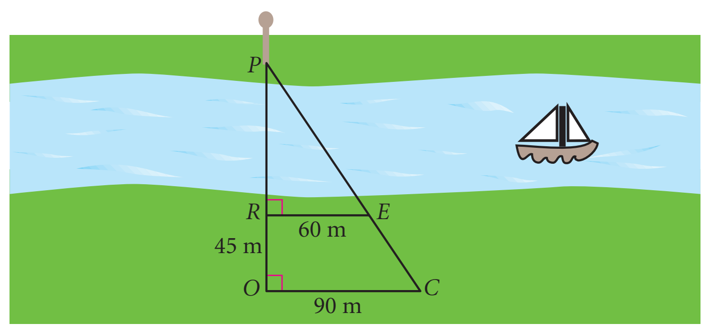
5. **APPLICATION** Juanita, who is 1.82 meters tall, wants to find the height of a tree in her backyard. From the tree's base, she walks 12.20 meters along the tree's shadow to a position where the end of her shadow exactly overlaps the end of the tree's shadow. She is now 6.10 meters from the end of the shadows. How tall is the tree?



6. While vacationing in Egypt, the Greek mathematician Thales calculated the height of the Great Pyramid. According to legend, Thales placed a pole at the tip of the pyramid's shadow and used similar triangles to calculate its height. This involved some estimating since he was unable to measure the distance from directly beneath the height of the pyramid to the tip of the shadow. From the diagram, explain his method. Calculate the height of the pyramid from the information given in the diagram.

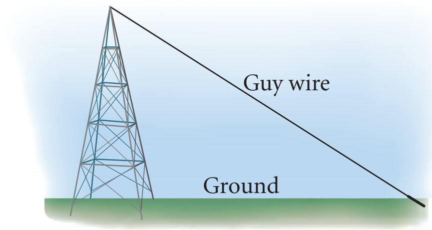


7. Calculate the distance across this river, PR, by sighting a pole, at point P, on the opposite bank. Points R and O are collinear with point P. Point C is chosen so that $\overline{OC} \perp \overline{PO}$. Lastly, point E is chosen so that P, E, and C are collinear and that $\overline{RE} \perp \overline{PO}$. Also explain why $\triangle PRE \sim \triangle POC$. ⓗ

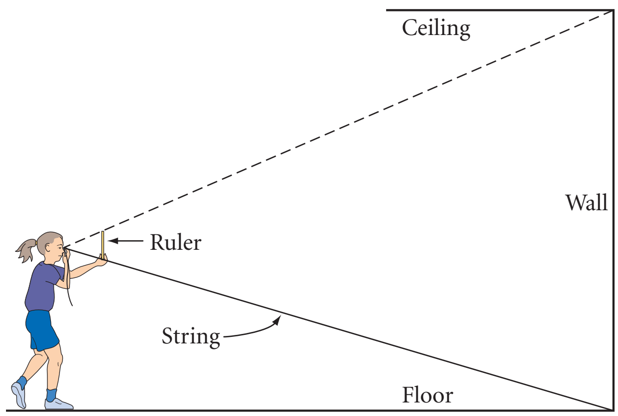


8. A pinhole camera is a simple device. Place unexposed film at one end of a shoe box, and make a pinhole at the opposite end. When light comes through the pinhole, an inverted image is produced on the film. Suppose you take a picture of a painting that is 30 cm wide by 45 cm high with a pinhole box camera that is 20 cm deep. How far from the painting should the pinhole be to make an image that is 2 cm wide by 3 cm high? Sketch a diagram of this situation. ⓗ

9. **APPLICATION** A guy wire attached to a high tower needs to be replaced. The contractor does not know the height of the tower or the length of the wire. Find a method to measure the length of the wire indirectly. ⓗ



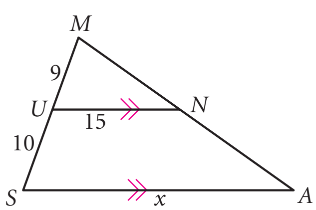
10. Kristin has developed a new method for indirectly measuring the height of her classroom. Her method uses string and a ruler. She tacks a piece of string to the base of the wall and walks back from the wall holding the other end of the string to her eye with her right hand. She holds a 12-inch ruler parallel to the wall in her left hand and adjusts her distance to the wall until the bottom of the ruler is in line with the bottom edge of the wall and the top of the ruler is in line with the top edge of the wall. Now with two measurements, she is able to calculate the height of the room. Explain her method. If the distance from her eye to the bottom of the ruler is 23 inches and the distance from her eye to the bottom of the wall is 276 inches, calculate the height of the room.



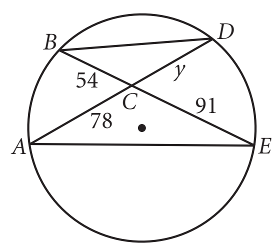
Review

For Exercises 11–13, first identify similar triangles and explain why they are similar. Then find the missing lengths.

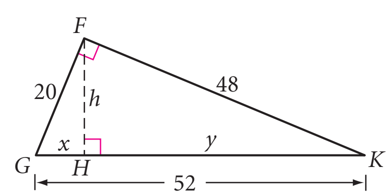
11. Find x. ⓗ



12. Find y.



13. Find x, y, and h. ⓗ



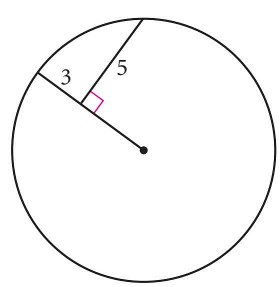
14. ***Construction*** Draw an obtuse triangle.

a. Use a compass and straightedge to construct two altitudes.

b. Use a ruler to measure both altitudes and their corresponding bases.

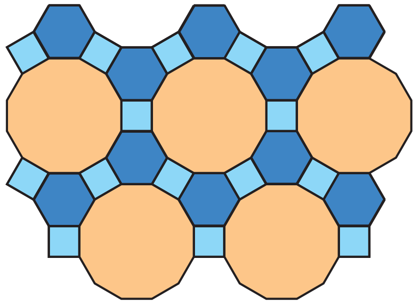
c. Calculate the area using both altitude-base pairs. Compare your results.

15. Find the radius of the circle.

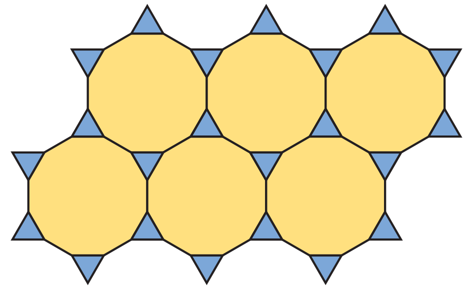


16. Give the vertex arrangement of each tessellation.

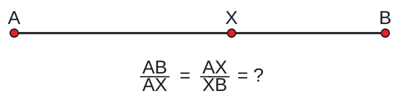
a.



b.



17. *Technology* On a segment AB, point X is called the **golden cut** if $\frac{AB}{AX} = \frac{AX}{XB}$, where $AX > XB$. The **golden ratio** is the value of $\frac{AB}{AX}$ and $\frac{AX}{XB}$ when they are equal. Use geometry software to explore the location of the golden cut on any segment AB. What is the value of the golden ratio? Find a way to construct the golden cut. ⓗ

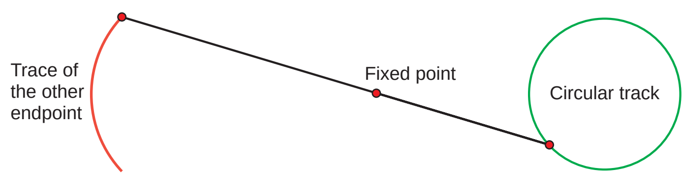


18. *Technology* Imagine that a rod of a given length is attached at one end to a circular track and passes through a fixed pivot point. As one endpoint moves around the circular track, the other endpoint traces a curve.

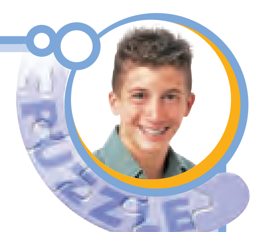
a. Predict what type of curve will be traced.

b. Model this situation with geometry software. Describe the curve that is traced.

c. Experiment with changing the size of the circular track, the length of the rod, or the location of the pivot point. Describe your results.

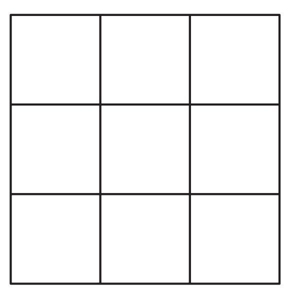


IMPROVING YOUR VISUAL THINKING SKILLS



TIC-TAC-NO!

Is it possible to shade six of the nine squares of a 3-by-3 grid so that no three of the shaded squares are in a straight line (row, column, or diagonal)?



LESSON

11.4 Corresponding Parts of Similar Triangles

Big doesn't necessarily mean better. Sunflowers aren't better than violets.

EDNA FERBER

Is there more to similar triangles than just proportional sides and congruent angles? For example, are there relationships between corresponding altitudes, corresponding medians, or corresponding angle bisectors in similar triangles? Let's investigate.

Investigation 1
Corresponding Parts

You will need
- a compass
- a straightedge

Use unlined paper for this investigation.

Step 1 Draw any triangle and construct a triangle of a different size similar to it. State the scale factor you used.

Step 2 Construct a pair of corresponding altitudes and use your compass to compare their lengths. How do they compare? How does the comparison relate to the scale factor you used?

Step 3 Construct a pair of corresponding medians. How do their lengths compare?

Step 4 Construct a pair of corresponding angle bisectors. How do their lengths compare?

Step 5 Compare your results with the results of others near you. You should be ready to make a conjecture.

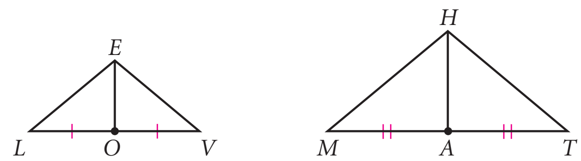
Proportional Parts Conjecture C-96

If two triangles are similar, then the corresponding _?_, _?_, and _?_ are _?_ to the corresponding sides.

The discovery you made in the investigation probably seems very intuitive. Let's see how you can prove one part of your conjecture. You will prove the other two parts in the exercises.

EXAMPLE Prove that corresponding medians of similar triangles are proportional to corresponding sides.

▸ Solution



Consider similar triangles $\triangle LVE$ and $\triangle MTH$ with corresponding medians $\overline{EO}$ and $\overline{HA}$. You need to show that the corresponding medians are proportional to corresponding sides, for example $\frac{EO}{HA} = \frac{EL}{HM}$. If you show that $\triangle LOE \sim \triangle MAH$ then you can show that $\frac{EO}{HA} = \frac{EL}{HM}$.

If you accept the SAS Similarity Conjecture as true, then you can show that $\triangle LOE \sim \triangle MAH$. You already know that $\angle L \cong \angle M$. Use algebra to show that $\frac{EL}{HM} = \frac{LO}{MA}$.

$$\frac{EL}{HM} = \frac{LV}{MT}$$ Corresponding sides of similar triangles $\triangle LVE$ and $\triangle MTH$ are proportional.

$$\frac{EL}{HM} = \frac{LO + OV}{MA + AT}$$ $LV = LO + OV$ and $MT = MA + AT$. Substitute.

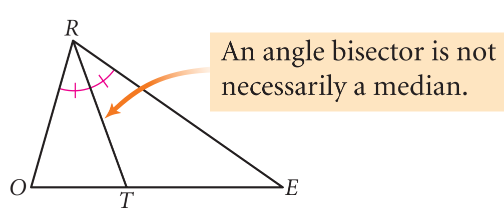
$$\frac{EL}{HM} = \frac{LO + LO}{MA + MA}$$ Since $\overline{EO}$ and $\overline{HA}$ are medians, O and A are midpoints. Since O and A are midpoints, $OV = LO$ and $AT = MA$. Substitute.

$$\frac{EL}{HM} = \frac{2LO}{2MA}$$ Add.

$$\frac{EL}{HM} = \frac{LO}{MA}$$ Reduce.

So, $\triangle LOE \sim \triangle MAH$ by the SAS Similarity Conjecture. Therefore you can also set up the proportion $\frac{EO}{HA} = \frac{EL}{HM}$, which shows that the corresponding medians are proportional to corresponding sides.

Recall when you first saw an angle bisector in a triangle. You may have thought that the bisector of an angle in a triangle divides the opposite side into two equal parts as well. A counterexample shows that this is not necessarily true. In $\triangle ROE$, $\overline{RT}$ bisects $\angle R$, but point T does not bisect $\overline{OE}$.



$OT \neq TE$

The angle bisector does, however, divide the opposite side in a particular way.

Investigation 2
Opposite Side Ratios

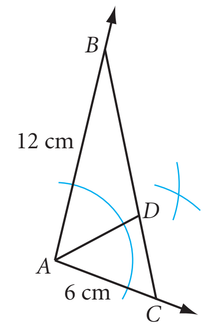
You will need

- a compass
- a ruler

In this investigation you'll discover that there is a proportional relationship involving angle bisectors.

Step 1 Draw any angle. Label it A.

Step 2 On one ray, locate point C so that AC is 6 cm. Use the same compass setting and locate point B on the other ray so that AB is 12 cm. Draw $\overline{BC}$ to form $\triangle ABC$.



Step 3 Construct the bisector of $\angle A$. Locate point D where the bisector intersects side $\overline{BC}$.

Step 4 Measure and compare CD and BD.

Step 5 Calculate and compare the ratios $\frac{CA}{BA}$ and $\frac{CD}{BD}$.

Step 6 Repeat Steps 1–5 with $AC = 10$ cm and $AB = 15$ cm.

Step 7 Compare your results with the results of others near you. State a conjecture.

Angle Bisector/Opposite Side Conjecture C-97

A bisector of an angle in a triangle divides the opposite side into two segments whose lengths are in the same ratio as __?__.

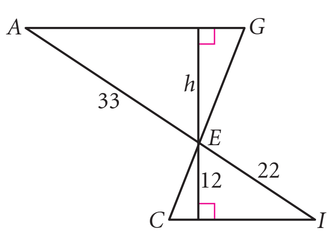
Exercises

You will need

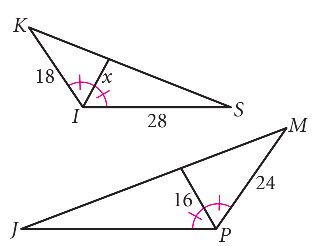
Construction tools for Exercises **16** and **23**

For Exercises 1–13, use your new conjectures. All measurements are in centimeters.

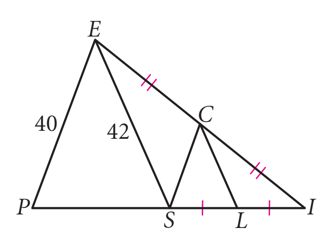
1. $\triangle ICE \sim \triangle AGE$
$h =$ __?__



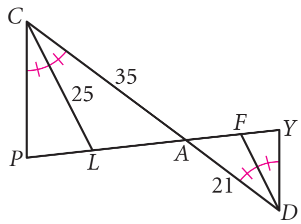
2. $\triangle SKI \sim \triangle JMP$
$x =$ __?__



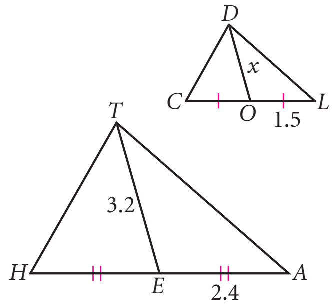
3. $\triangle PIE \sim \triangle SIC$
Point S is the midpoint of PI.
$CL =$ __?__, $CS =$ __?__ ⓗ



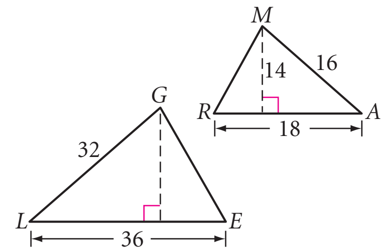
4. $\triangle CAP \sim \triangle DAY$
$FD = \underline{\ ?\ }$



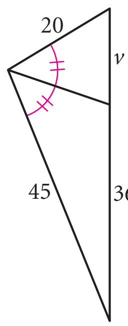
5. $\triangle HAT \sim \triangle CLD$
$x = \underline{\ ?\ }$



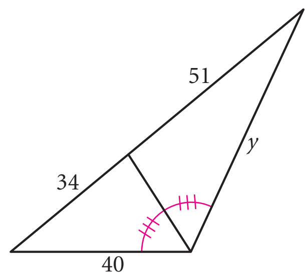
6. $\triangle ARM \sim \triangle LEG$
Area of $\triangle ARM = \underline{\ ?\ }$
Area of $\triangle LEG = \underline{\ ?\ }$



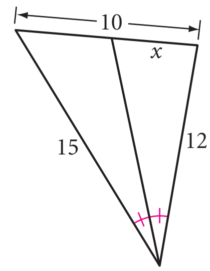
7. $v = \underline{\ ?\ }$



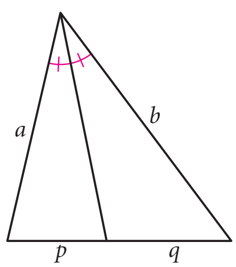
8. $y = \underline{\ ?\ }$



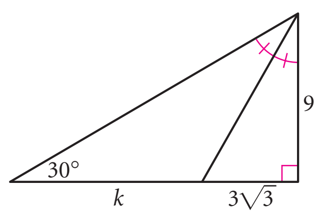
9. $x = \underline{\ ?\ }$ ⓗ



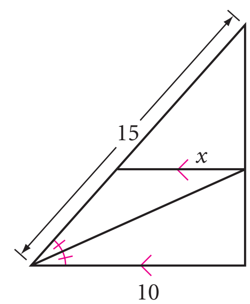
10. $\frac{a}{b} = \underline{\ ?\ }, \frac{a}{p} = \underline{\ ?\ }$



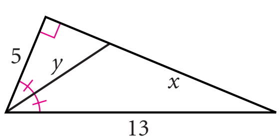
11. $k = \underline{\ ?\ }$



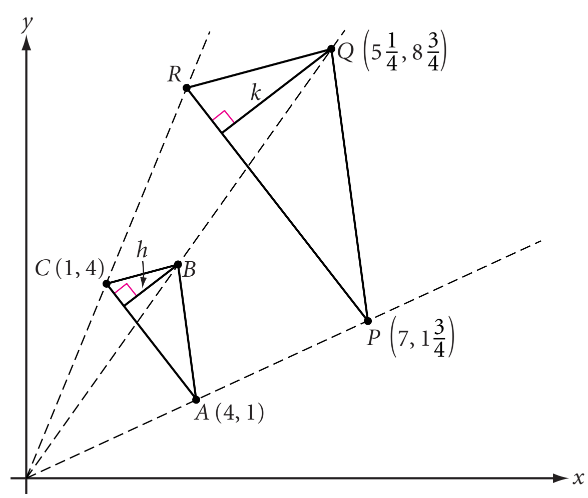
12. $x = \underline{\ ?\ }$ ⓗ



13. $x = \underline{\ ?\ }, y = \underline{\ ?\ }$



14. Triangle PQR is the image of $\triangle ABC$ under a dilation. Find the coordinates of B and R. Find the ratio $\frac{k}{h}$.



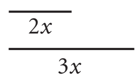
15. Aunt Florence has willed to her two nephews a plot of land in the shape of an isosceles right triangle. The land is to be divided into two unequal parts by bisecting one of the two congruent angles. What is the ratio of the greater area to the lesser area?

16. ***Construction*** How would you divide a segment into lengths with a ratio of $\frac{2}{3}$? The Angle Bisector/Opposite Side Conjecture gives you a way to do this. To get you started, here are the first four steps. ⓗ

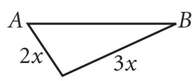
Step 1 Construct any segment AB.

Step 2 Construct a second segment. Call its length x.

Step 3 Construct two more segments with lengths $2x$ and $3x$.



Step 4 Construct a triangle with lengths $2x$, $3x$, and AB.



You're on your own from here!

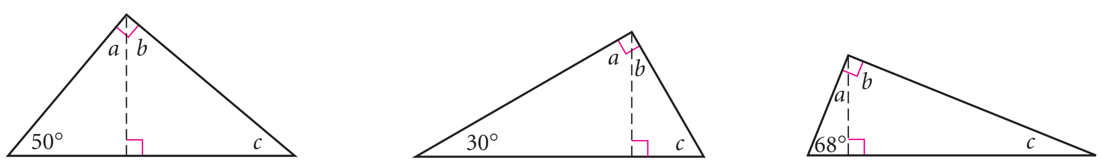
17. Prove that corresponding angle bisectors of similar triangles are proportional to corresponding sides. ⓗ

18. Prove that corresponding altitudes of similar triangles are proportional to corresponding sides.

19. ***Mini-Investigation*** This investigation is in two parts. You will need to complete the conjecture in part a before moving on to part b.

a. The altitude to the hypotenuse has been constructed in each right triangle below. This construction creates two smaller right triangles within each original right triangle. Calculate the measures of the acute angles in each diagram.

i. $a = \underline{?}$, $b = \underline{?}$, $c = \underline{?}$ **ii.** $a = \underline{?}$, $b = \underline{?}$, $c = \underline{?}$ **iii.** $a = \underline{?}$, $b = \underline{?}$, $c = \underline{?}$

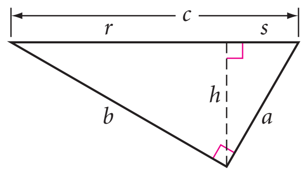


How do the smaller right triangles compare in each diagram? How do they compare to the original right triangle? You should be ready to state a conjecture.

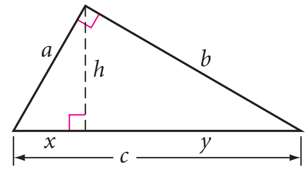
Conjecture: The altitude to the hypotenuse of a right triangle divides the triangle into two right triangles that are $\underline{?}$ to each other and to the original $\underline{?}$.

b. Complete each proportion for these right triangles.

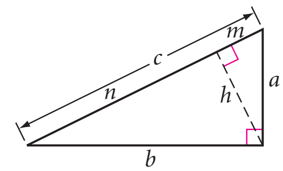
i. $\frac{h}{r} = \frac{s}{?}$



ii. $\frac{y}{h} = \frac{?}{x}$



iii. $\frac{n}{h} = \frac{?}{?}$



Review the proportions you wrote. How are they alike? You should be ready to state a conjecture.

Conjecture: The altitude (length h) to the hypotenuse of a right triangle divides the hypotenuse into two segments (lengths p and q), such that $\frac{p}{?} = \frac{?}{q}$.

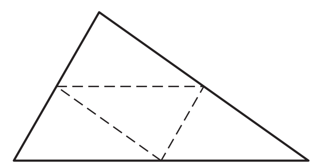
Add these conjectures to your notebook.

Review

20. Use algebra to show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$. ⓗ

21. A rectangle is divided into four rectangles, each similar to the original rectangle. What is the ratio of short side to long side in the rectangles?

22. In Chapter 5, you discovered that when you construct the three midsegments in a triangle, they divide the triangle into four congruent triangles. Are the four triangles similar to the original? Explain why.



23. *Construction* Draw any triangle ABC. Select any point X on $\overline{AB}$. Construct a line through X parallel to $\overline{AC}$ that intersects $\overline{BC}$ in point Y. Find a proportion that relates AX, XB, BY, and YC.

24. A rectangle has sides a and b. For what values of a and b is another rectangle with sides $2a$ and $\frac{b}{2}$

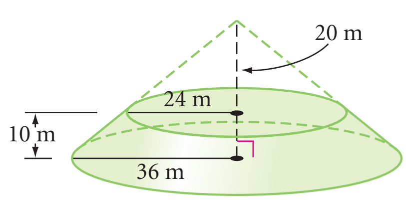
a. Congruent to the original?

b. Equal in perimeter to the original?

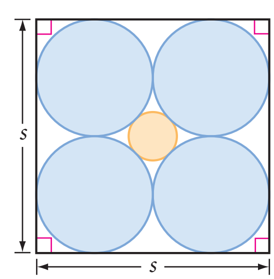
c. Equal in area to the original?

d. Similar but not congruent to the original?

25. Find the volume of this truncated cone.



26. The large circles are tangent to the square and tangent to each other. The smaller circle is tangent to each larger circle. Find the radius of the smaller circle in terms of s, the length of each side of the square. ⓗ



LESSON

11.5

Proportions with Area and Volume

It is easy to show that a hare could not be as large as a hippopotamus, or a whale as small as a herring. For every type of animal there is a most convenient size, and a large change in size inevitably carries with it a change of form.

J. B. S. HALDANE

You can use similarity to find the surface areas and volumes of objects that are geometrically similar. Suppose an artist wishes to gold-plate a sculpture. If it costs \$250 to gold-plate a model that is half as long in each dimension, how much will it cost for the full-size sculpture? Not \$500, but \$1000! If the model weighs 40 pounds, how much will the full-size sculpture weigh? Not 80 pounds, but 320 pounds! In this lesson you will discover why these answers may not be what you expected.

You might recognize this giant golden man as the Academy Awards statuette. Also called the Oscar, smaller versions of the figure are handed out annually for excellence in the motion picture industry. If this statuette were real gold, it would be very expensive and incredibly heavy.

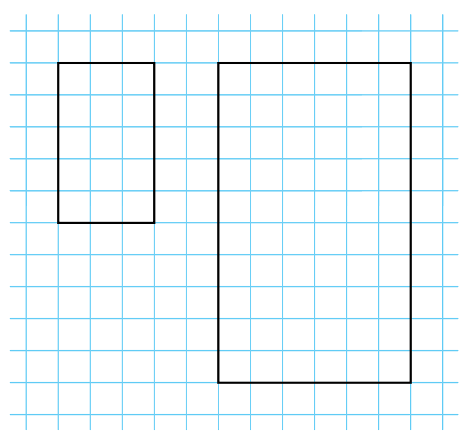
Investigation 1

Area Ratios

You will need

- graph paper

In this investigation you will find the relationship between areas of similar figures.



Step 1 Draw a rectangle on graph paper. Calculate its area.

Step 2 Draw a rectangle similar to your first rectangle by multiplying its sides by a scale factor. Calculate this area.

Step 3 What is the ratio of side lengths (larger to smaller) for your two rectangles? What is the ratio of their areas (larger to smaller)?

Step 4 How many copies of the smaller rectangle would you need to fill the larger rectangle? Draw lines in your larger rectangle to show how you would place the copies to fill the area.

Step 5 Compare your results with the results of others near you.

Step 6 Repeat Steps 1–5 using triangles instead of rectangles.

Step 7 Discuss whether or not your findings would apply to any pair of similar polygons. Would they apply to similar circles or other curved figures? You should be ready to state a conjecture.

Proportional Areas Conjecture

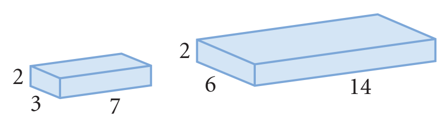
C-98

If corresponding sides of two similar polygons or the radii of two circles compare in the ratio $\frac{m}{n}$, then their areas compare in the ratio _?_.

Similar solids are solids that have the same shape but not necessarily the same size. All cubes are similar, but not all prisms are similar. All spheres are similar, but not all cylinders are similar. Two polyhedrons are similar if all their corresponding faces are similar and the lengths of their corresponding edges are proportional. Two right cylinders (or right cones) are similar if their radii and heights are proportional.

EXAMPLE A

Are these right rectangular prisms similar?



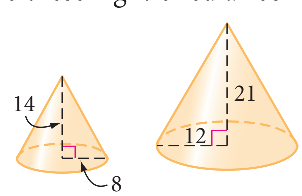
▶ Solution

The two prisms are not similar because the corresponding edges are not proportional.

$$\frac{2}{2} \neq \frac{3}{6} = \frac{7}{14}$$

EXAMPLE B

Are these right circular cones similar?



▶ Solution

The two cones are similar because the radii and heights are proportional.

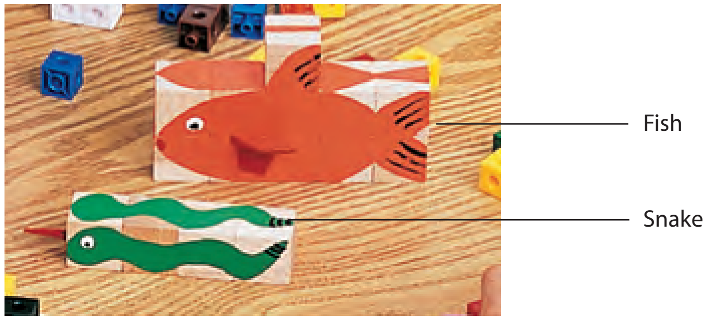
$$\frac{8}{12} = \frac{14}{21}$$

Investigation 2
Volume Ratios

You will need

- interlocking cubes

How does the ratio of lengths of corresponding edges of similar solids compare with the ratio of their volumes? Let's find out.



Step 1 Use blocks to build the "snake." Calculate its volume.

Step 2 Build a similar snake by multiplying every dimension by a scale factor of 3. Calculate this volume.

Step 3 What is the ratio of side lengths (larger to smaller) for your two snakes? What is the ratio of volumes (larger to smaller)?

Step 4 As in Steps 1–3, use blocks to build the "fish" and another fish similar to it, this time by a scale factor of 2. Find the ratio of the side lengths and the ratio of the volumes.

Step 5 How do your results compare with the results in Investigation 1? Discuss how you would calculate the volume of a snake increased by a scale factor of 5. Discuss how you would calculate the volume of a fish increased by a scale factor of 4.

Step 6 You should be ready to state a conjecture.

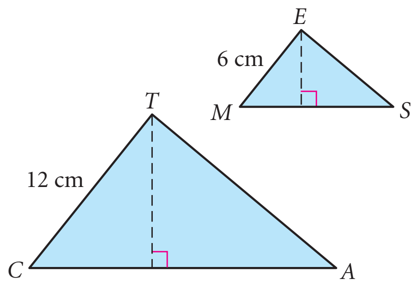
Proportional Volumes Conjecture

C-99

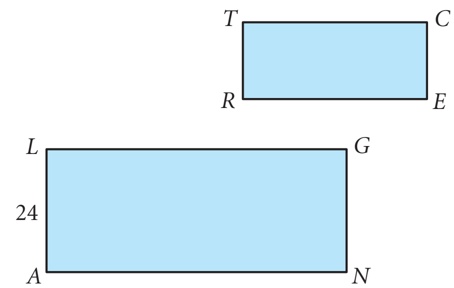
If corresponding edges (or radii, or heights) of two similar solids compare in the ratio $\frac{m}{n}$, then their volumes compare in the ratio $\underline{\ ?\ }$.

EXERCISES

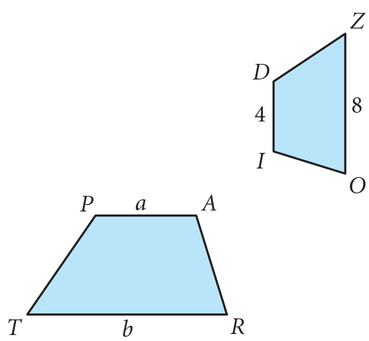
1. $\triangle CAT \sim \triangle MSE$
Area of $\triangle CAT = 72\ \text{cm}^2$
Area of $\triangle MSE = \underline{\ ?\ }$ ⓗ



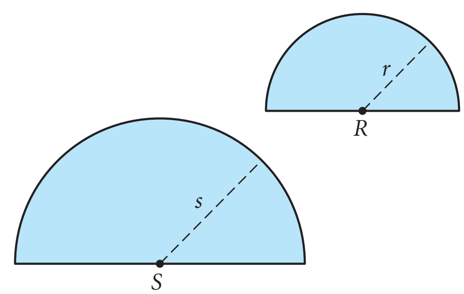
2. $RECT \sim ANGL$
$$\frac{\text{Area of } RECT}{\text{Area of } ANGL} = \frac{9}{16}$$
$TR = \underline{\ ?\ }$



3. $TRAP \sim ZOID$ ⓗ
$$\frac{\text{Area of } ZOID}{\text{Area of } TRAP} = \frac{16}{25}$$
$a = \underline{\ ?\ }$, $b = \underline{\ ?\ }$



4. semicircle $R \sim$ semicircle S
$$\frac{r}{s} = \frac{3}{5}$$
Area of semicircle $S = 75\pi\ \text{cm}^2$
Area of semicircle $R = \underline{\ ?\ }$



5. The ratio of the lengths of corresponding diagonals of two similar kites is $\frac{1}{7}$. What is the ratio of their areas?

6. The ratio of the areas of two similar trapezoids is $\frac{1}{9}$. What is the ratio of the lengths of their altitudes?

7. The ratio of the lengths of the edges of two cubes is $\frac{m}{n}$. What is the ratio of their surface areas? ⓗ

8. The celestial sphere shown at right has radius 9 inches. The planet in the sphere's center has radius 3 inches. What is the ratio of the volume of the planet to the volume of the celestial sphere? What is the ratio of the surface area of the planet to the surface area of the celestial sphere?

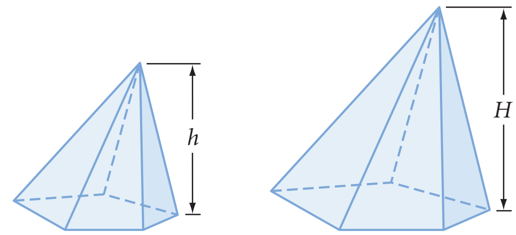
9. **APPLICATION** Annie works in a magazine's advertising department. A client has requested that his 5 cm-by-12 cm ad be enlarged: "Double the length and double the width, then send me the bill." The original ad cost \$1500. How much should Annie charge for the larger ad? Explain your reasoning.

10. The pentagonal pyramids are similar.

$\frac{h}{H} = \frac{4}{7}$

Volume of large pyramid $= \underline{\ ?\ }$

Volume of small pyramid $= 320\ \text{cm}^3$



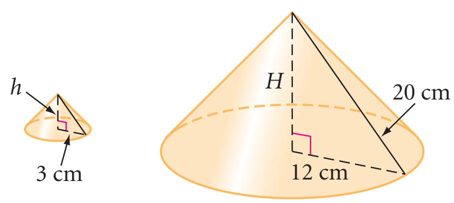
11. These right cones are similar.

$H = \underline{\ ?\ }$, $h = \underline{\ ?\ }$

Volume of large cone $= \underline{\ ?\ }$

Volume of small cone $= \underline{\ ?\ }$

$\frac{\text{Volume of large cone}}{\text{Volume of small cone}} = \underline{\ ?\ }$



12. These right trapezoidal prisms are similar.

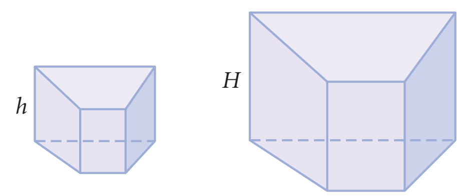
Volume of small prism $= 324\ \text{cm}^3$

$\frac{\text{Area of base of small prism}}{\text{Area of base of large prism}} = \frac{9}{25}$

$\frac{h}{H} = \underline{\ ?\ }$

$\frac{\text{Volume of large prism}}{\text{Volume of small prism}} = \underline{\ ?\ }$

Volume of large prism $= \underline{\ ?\ }$ ⓗ



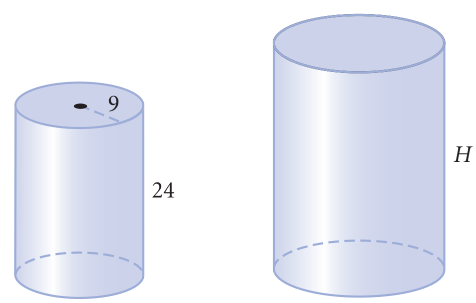
13. These right cylinders are similar.

Volume of large cylinder $= 4608\pi\ \text{ft}^3$

Volume of small cylinder $= \underline{\ ?\ }$

$\frac{\text{Volume of large cylinder}}{\text{Volume of small cylinder}} = \underline{\ ?\ }$

$H = \underline{\ ?\ }$



14. The ratio of the lengths of corresponding edges of two similar triangular prisms is $\frac{5}{3}$. What is the ratio of their volumes?

15. The ratio of the volumes of two similar pentagonal prisms is $\frac{8}{125}$. What is the ratio of their heights?

16. The ratio of the weights of two spherical steel balls is $\frac{8}{27}$. What is the ratio of their diameters?

17. **APPLICATION** The energy (and cost) needed to operate an air conditioner is proportional to the volume of the space that is being cooled. It costs ZAP Electronics about \$125 per day to run an air conditioner in their small rectangular warehouse. The company's large warehouse, a few blocks away, is 2.5 times as long, wide, and high as the small warehouse. Estimate the daily cost of cooling the large warehouse with the same model of air conditioner. ⓗ

18. **APPLICATION** A sculptor creates a small bronze statue that weighs 38 lb. She plans to make a version that will be four times as large in each dimension. How much will this larger statue weigh if it is also bronze?

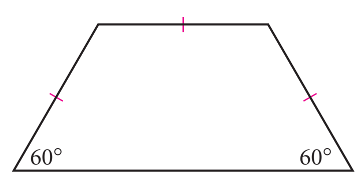
This bronze sculpture by Camille Claudel (1864–1943) is titled *La Petite Chatelaine*. Claudel was a notable French artist and student of Auguste Rodin, whose famous sculptures include *The Thinker*.

19. A tabloid magazine at a supermarket checkout exclaims, "Scientists Breed 4-Foot Tall Chicken." A photo shows a giant chicken that supposedly weighs 74 pounds and will solve the world's hunger problem. What do you think about this headline? Assuming an average chicken stands 14 inches tall and weighs 7 pounds, would a 4-foot chicken weigh 74 pounds? Is it possible for a chicken to be 4 feet tall? Explain your reasoning.

20. The African goliath frog shown in this photo is the largest known frog—about 0.3 m long and 3.2 kg in weight. The Brazilian gold frog is one of the smallest known frogs—about 9.8 mm long. Approximate the weight of a gold frog. What assumptions do you need to make? Explain your reasoning.

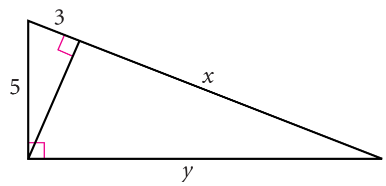
Review

21. Make four copies of the trapezoid at right. Arrange them into a similar but larger trapezoid. Sketch the final trapezoid and show how the smaller trapezoids fit inside it.

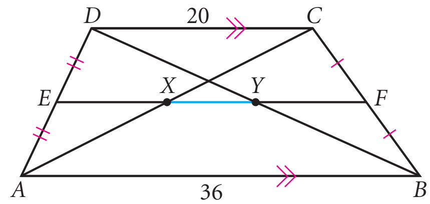


22. Sara rents her goat, Munchie, as a lawn mower. Munchie is tied to a stake with a 10 m rope. Sara wants to find an efficient pattern for Munchie's stake positions so that all grass in a field 42 m by 42 m is mowed but overlap is minimized. Make a sketch showing all the stake positions needed.

23. $x = \underline{\ ?\ }$, $y = \underline{\ ?\ }$



24. $XY = \underline{\ ?\ }$

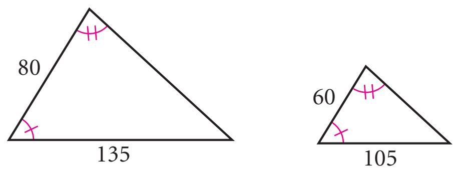


25. Find the area of a regular decagon with an apothem 5.7 cm and a perimeter 37 cm.

26. Find the area of a triangle whose sides measure 13 feet, 13 feet, and 10 feet. ⓗ

27. True or false? Every cross section of a pyramid has the same shape as, but a different size from the base. ⓗ

28. What's wrong with this picture?



project

IN SEARCH OF THE PERFECT RECTANGLE

A square is a perfectly symmetric quadrilateral. Yet, people rarely make books, posters, or magazines that are square. Instead, most people seem to prefer rectangles. In fact, some people believe that a particular type of rectangle is more appealing because its proportions fit the golden ratio. You can learn more about the historical importance of the golden ratio by doing research with the links at www.keymath.com/DG .

Do people have a tendency to choose a particular length/width ratio when they design or build common objects? Find at least ten different rectangular objects in your classroom or home: books, postcards, desks, doors, and other everyday items. Measure the longer side and the shorter side of each one. Predict what a graph of your data will look like.

Now graph your data. Is there a pattern? Find the line of best fit. How well does it fit the data? What is the range of length/width ratios? What ratio do points on the line of best fit represent?

Your project should include

- A table and graph of your data.
- Your predictions and your analysis.
- A paragraph explaining your opinion about whether or not people have a tendency to choose a particular type of rectangle and why.

Fathom™

You can use Fathom to graph your data and find the line of best fit. Choose different types of graphs to get other insights into what your data mean.

Exploration

Why Elephants Have Big Ears

The relationship between surface area and volume is of critical importance to all living things. It explains why elephants have big ears, why hippos and rhinos have short, thick legs and must spend a lot of time in water, and why movie monsters like King Kong and Godzilla can't exist.

Activity

Convenient Sizes

Body Temperature

Every living thing processes food for energy. This energy creates heat that radiates from its surface.

Step 1 Imagine two similar animals, one with dimensions three times as large as those of the other.

Step 2 How would the surface areas of these two animals compare? How much more heat could the larger animal radiate through its surface?

Step 3 How would the volumes of these two animals compare? If the animals' bodies produce energy in proportion to their volumes, how many times as much heat would the larger animal produce?

Step 4 Review your answers from Steps 2 and 3. How many times as much heat must each square centimeter of the larger animal radiate? Would this be good or bad?

Step 5 Use what you have concluded to answer these questions. Consider size, surface area, and volume.

a. Why do large objects cool more slowly than similar small objects?

b. Why is a beached whale more likely than a beached dolphin to experience overheating?

c. Why are larger mammals found closer to the poles than the equator?

d. If a woman and a small child fall into a cold lake, why is the child in greater danger of hypothermia?

e. When the weather is cold, iguanas hardly move. When it warms up, they become active. If a small iguana and a large iguana are sunning themselves in the morning sun, which one will become active first? Why? Which iguana will remain active longer after sunset? Why?

f. Why do elephants have big ears?

Bone and Muscle Strength

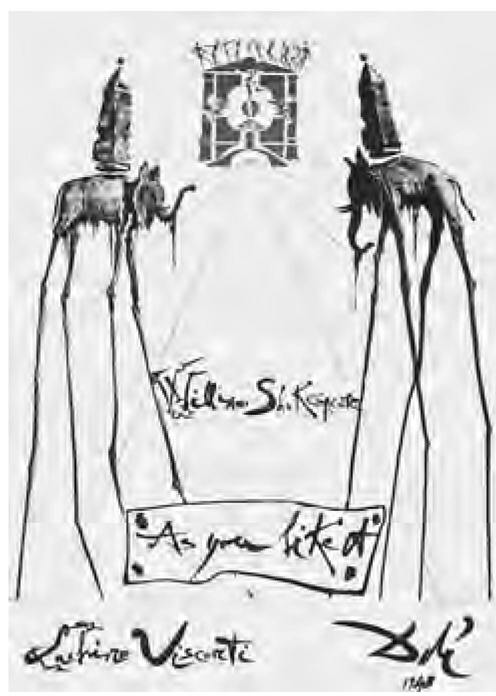
The strength of a bone or a muscle is proportional to its cross-sectional area.

Step 6 Imagine a 7-foot-tall basketball player and a 42-inch-tall child. How would the dimensions of the bones of the basketball player compare to the corresponding bones of the child?

Step 7 How would the cross-sectional areas of their corresponding bones compare?

Step 8 How would their weights compare?

Step 9 How many times as much weight would each cross-sectional square inch of bone have to support in the basketball player? What are some factors that may explain why basketball players' bones don't usually break?



The Spanish artist Salvador Dalí (1904–1989) designed this cover for a 1948 program of the ballet *As You Like It.* What is the effect created by the elephants on long spindly legs? Could these animals really exist? Explain.

Step 10

Use what you have concluded to answer these questions. Consider size, surface area, and volume.

a. Why are the largest living mammals, the whales, confined to the sea?

b. Why do hippos and rhinos have short, thick legs?

c. Why are champion weight lifters seldom able to lift more than twice their weight?

d. Thoroughbred racehorses are fast runners but break their legs easily, while draft horses are slow moving and rarely break their legs. Why is this?

e. Assume that a male gorilla can weigh as much as 450 pounds and can reach about 6 feet tall. King Kong is 30 feet tall. Could King Kong really exist?

f. Professional basketball players are not typically similar in shape to professional football players. Discuss the advantages and disadvantages of each body type in each sport.

Similarity is a theme in *King Kong* (1933), a movie about a giant gorilla similar in shape to a real gorilla. Willis O'Brien (1886–1962), an innovator of stop-motion animation, designed the models of Kong that brought the gorilla to life for moviegoers everywhere. Here, special effects master Ray Harryhausen holds a model that was used to create this captivating scene.

Gravity and Air Resistance

Objects in a vacuum fall at the same rate. However, an object falling through air is slowed by air resistance. Air resistance is proportional to the surface area of the falling object.

Step 11

Imagine a rat that is 8 times the length, width, and height of a similar mouse. Both animals fall from a cliff.

Step 12

How would the volumes of the two animals compare?

Step 13 How would the air resistance against the two animals compare?

Step 14 The mass (related to weight) of an object is a factor in the force of the impact of the object with the ground. But air resistance on an object slows its fall, counteracting some of the force of impact. Assume that the weights of the rat and mouse are proportional to their volumes. Compare the force of the rat's impact with the ground to the force of the mouse's impact. Which is more likely to survive the fall?

Step 15 Use what you have concluded to answer this question. Consider size, surface area, and volume.

An ant can fall 100 times its height and live. This is not true for a human. Why?

IMPROVING YOUR VISUAL THINKING SKILLS

Painted Faces II

Small cubes are assembled to form a larger cube, and then some of the faces of this larger cube are painted. After the paint dries, the larger cube is taken apart. Exactly 60 of the small cubes have no paint on any of their faces. What were the dimensions of the larger cube? How many of its faces were painted?

LESSON 11.6

Proportional Segments Between Parallel Lines

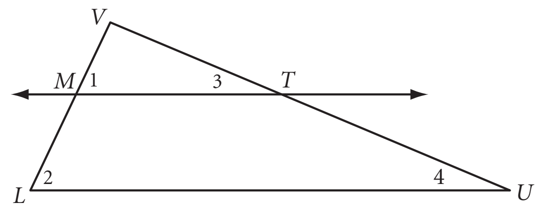
Mistakes are the portals of discovery.

JAMES JOYCE

In the figure below, $\overleftrightarrow{MT} \parallel \overline{LU}$. Is $\triangle LUV$ similar to $\triangle MTV$? Yes, it is. A short paragraph proof can support this observation.

Given: $\triangle LUV$ with $\overleftrightarrow{MT} \parallel \overline{LU}$

Show: $\triangle LUV \sim \triangle MTV$



Paragraph Proof

First assume that the Corresponding Angles Conjecture and the AA Similarity Conjecture are true.

If $\overleftrightarrow{MT} \parallel \overline{LU}$, then $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$ by the Corresponding Angles Conjecture.

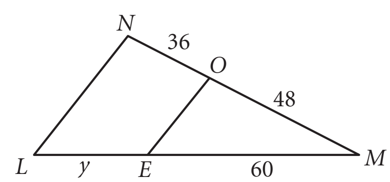
If $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$, then $\triangle LUV \sim \triangle MTV$ by the AA Similarity Conjecture. ■

Let's see how you can use this observation to solve problems.

EXAMPLE A

$\overline{EO} \parallel \overline{LN}$

$y = \underline{\ ?\ }$



▸ Solution

Use the fact that $\triangle EMO \sim \triangle LMN$ to write a proportion with the lengths of corresponding sides.

$\frac{MO}{MN} = \frac{ME}{ML}$	Corresponding sides of similar triangles are proportional.
$\frac{48}{48+36} = \frac{60}{60+y}$	Substitute lengths given in the figure.
$\frac{4}{7} = \frac{60}{60+y}$	Reduce the left side of the equation.
$240 + 4y = 420$	Multiply both sides by $7(60 + y)$, reduce, and distribute.
$4y = 180$	Subtract 240 from both sides.
$y = 45$	Divide by 4.

Look back at the figure in Example A. Notice that the ratio $\frac{LE}{EM}$ is the same as the ratio $\frac{NO}{OM}$. So there are more relationships in the figure than the ones we find in similar triangles. Let's investigate.

Investigation 1
Parallels and Proportionality

You will need

- a ruler
- a protractor

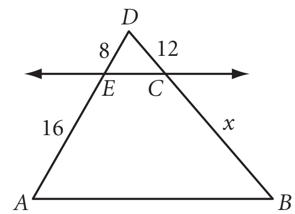
In this investigation, we'll look at the ratios of segments that have been cut by parallel lines.

Step 1 In each figure below, find x. Then find numerical values for the ratios.

a. $\overleftrightarrow{EC} \parallel \overline{AB}$

$x = \underline{?}$

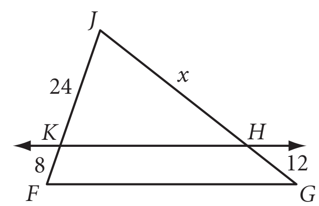
$\frac{DE}{AE} = \underline{?}, \frac{DC}{BC} = \underline{?}$



b. $\overleftrightarrow{KH} \parallel \overline{FG}$

$x = \underline{?}$

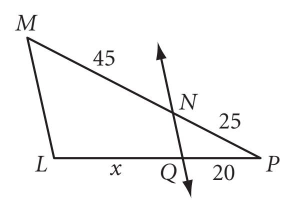
$\frac{JK}{KF} = \underline{?}, \frac{JH}{HG} = \underline{?}$



c. $\overleftrightarrow{QN} \parallel \overline{LM}$

$x = \underline{?}$

$\frac{PQ}{QL} = \underline{?}, \frac{PN}{MN} = \underline{?}$



Step 2 What do you notice about the ratios of the lengths of the segments that have been cut by the parallel lines?

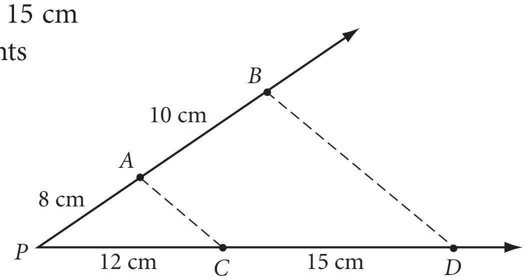
Is the converse true? That is, if a line divides two sides of a triangle proportionally, is it parallel to the third side? Let's see.

Step 3 Draw an acute angle, P.

Step 4 Beginning at point P, use your ruler to mark off lengths of 8 cm and 10 cm on one ray. Label the points A and B.

Step 5 Mark off lengths of 12 cm and 15 cm on the other ray. Label the points C and D. Notice that $\frac{8}{10} = \frac{12}{15}$.

Step 6 Draw $\overline{AC}$ and $\overline{BD}$.



Step 7 With a protractor, measure $\angle PAC$ and $\angle PBD$. Are $\overline{AC}$ and $\overline{BD}$ parallel?

Step 8 Repeat Steps 3–7, but this time use your ruler to create your own lengths such that $\frac{PA}{AB} = \frac{PC}{CD}$.

Step 9 Compare your results with the results of others near you.

You should be ready to combine your observations from Steps 2 and 9 into one conjecture.

Parallel/Proportionality Conjecture **C-100**

If a line parallel to one side of a triangle passes through the other two sides, then it divides the other two sides __?__. Conversely, if a line cuts two sides of a triangle proportionally, then it is __?__ to the third side.

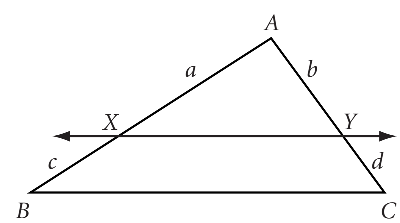
If you assume that the AA Similarity Conjecture is true, you can use algebra to prove the Parallel/Proportionality Conjecture. Here's the first part.

EXAMPLE B

Given: $\triangle ABC$ with $\overleftrightarrow{XY} \parallel \overline{BC}$

Show: $\frac{a}{c} = \frac{b}{d}$

(Assume that the lengths a, b, c, and d are all nonzero.)



▶ Solution

First, you know that $\triangle AXY \sim \triangle ABC$ (see the proof on page 603). Use a proportion of corresponding sides.

$$\frac{a}{a+c} = \frac{b}{b+d}$$ Lengths of corresponding sides of similar triangles are proportional.

$$\frac{a(a+c)(b+d)}{(a+c)} = \frac{b(a+c)(b+d)}{(b+d)}$$ Multiply both sides by $(a+c)(b+d)$.

$$a(b+d) = b(a+c)$$ Reduce.

$$ab + ad = ba + bc$$ Apply the distributive property.

$$ab + ad = ab + bc$$ Commute ba to ab.

$$ad = bc$$ Subtract ab from both sides.

$$\frac{ad}{cd} = \frac{bc}{cd}$$ We want c and d in the denominator, so divide both sides by cd.

$$\frac{a}{c} = \frac{b}{d}$$ Reduce.

You'll prove the converse of the Parallel/Proportionality Conjecture in Exercise 18.

Can the Parallel/Proportionality Conjecture help you divide segments into several proportional parts? Let's investigate.

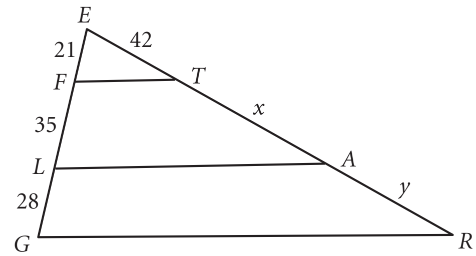
Investigation 2
Extended Parallel/Proportionality

Step 1 Use the Parallel/Proportionality Conjecture to find each missing length. Are the ratios equal?

a. $\overline{FT} \parallel \overline{LA} \parallel \overline{GR}$

$x = \underline{\ ?\ }$, $y = \underline{\ ?\ }$

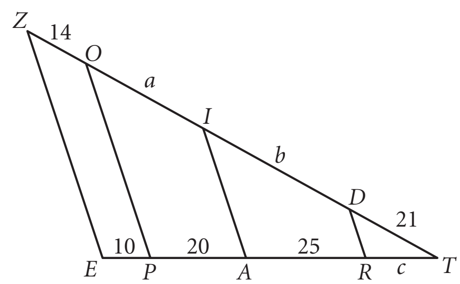
Is $\dfrac{FL}{LG} = \dfrac{TA}{AR}$?



b. $\overline{ZE} \parallel \overline{OP} \parallel \overline{IA} \parallel \overline{DR}$

$a = \underline{\ ?\ }$, $b = \underline{\ ?\ }$, $c = \underline{\ ?\ }$

Is $\dfrac{DI}{IO} = \dfrac{RA}{AP}$? Is $\dfrac{IO}{OZ} = \dfrac{AP}{PE}$?



Step 2 Compare your results with the results of others near you. Complete the conjecture below.

Extended Parallel/Proportionality Conjecture

C-101

If two or more lines pass through two sides of a triangle parallel to the third side, then they divide the two sides $\underline{\ ?\ }$.

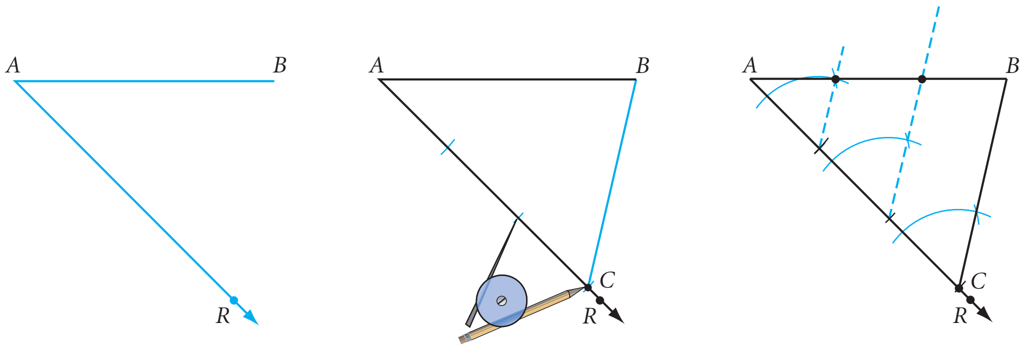
Exploring the converse of this conjecture has been left for you as a Take Another Look activity.

You already know how to use a perpendicular bisector to divide a segment into two, four, or eight equal parts. Now you can use your new conjecture to divide a segment into *any* number of equal parts.

EXAMPLE C Divide any segment AB into three congruent parts using only a compass and straightedge.

▶ Solution Draw segment AB. From one endpoint of $\overline{AB}$, draw any ray to form an angle. On the ray, mark off three congruent segments with your compass. Connect the third compass mark to the other endpoint of $\overline{AB}$ to form a triangle.

Finally, through the other two compass marks on the ray, construct lines parallel to the third side of the triangle. The two parallel lines divide $\overline{AB}$ into three equal parts.



EXERCISES

You will need

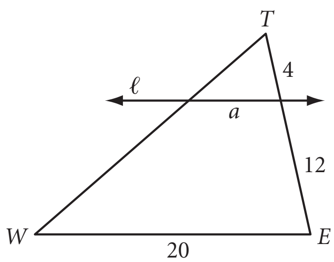
Construction tools for Exercises **13** and **14**

Geometry software for Exercise **26**

For Exercises 1–12, all measurements are in centimeters.

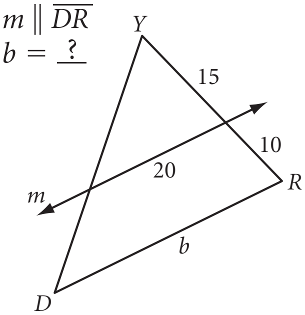
1. $\ell \parallel \overline{WE}$

$a = \underline{\ ?\ }$ ⓗ



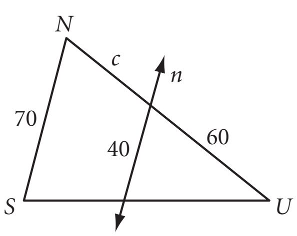
2. $m \parallel \overline{DR}$

$b = \underline{\ ?\ }$



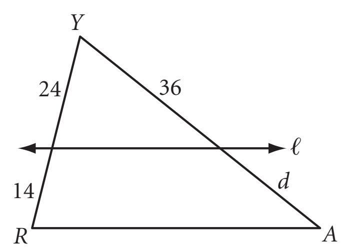
3. $n \parallel \overline{SN}$

$c = \underline{\ ?\ }$ ⓗ



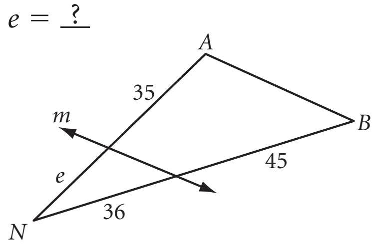
4. $\ell \parallel \overline{RA}$

$d = \underline{\ ?\ }$ ⓗ

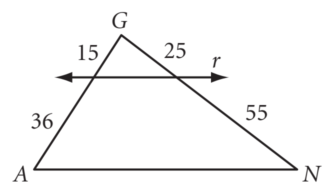


5. $m \parallel \overline{BA}$

$e = \underline{\ ?\ }$

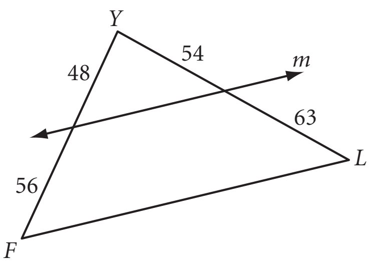


6. Is $r \parallel \overline{AN}$? ⓗ



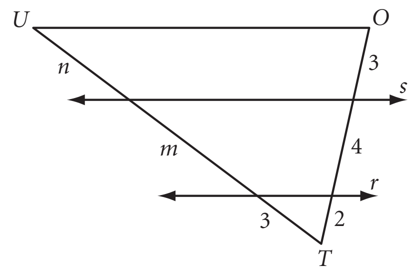
Similarity is used to create integrated circuits. Electrical engineers use large-scale maps of extremely small silicon chips. This engineer is making a scale drawing of a computer chip.

7. Is $m \parallel \overline{FL}$?



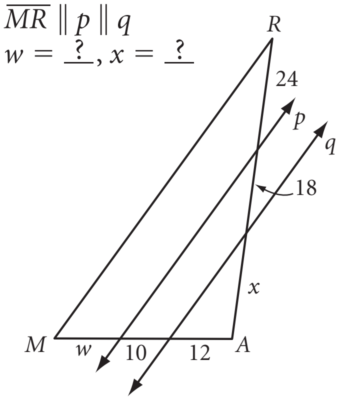
8. $r \parallel s \parallel \overline{OU}$

$m = \underline{\ ?\ }$, $n = \underline{\ ?\ }$



9. $\overline{MR} \parallel p \parallel q$

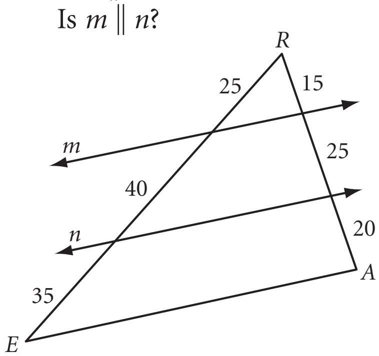
$w = \underline{\ ?\ }$, $x = \underline{\ ?\ }$



10. Is $m \parallel \overline{EA}$?

Is $n \parallel \overline{EA}$?

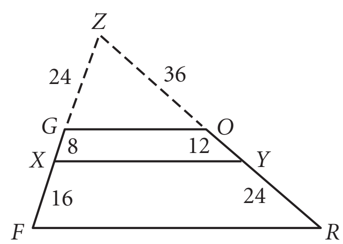
Is $m \parallel n$?



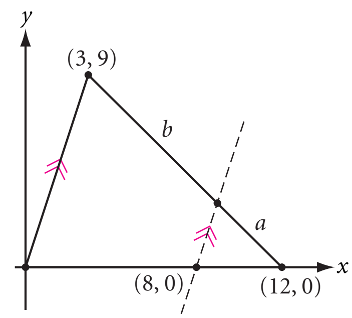
11. Is $\overline{XY} \parallel \overline{GO}$?

Is $\overline{XY} \parallel \overline{FR}$?

Is *FROG* a trapezoid?



12. $a = \underline{\ ?\ }$, $b = \underline{\ ?\ }$ ⓗ

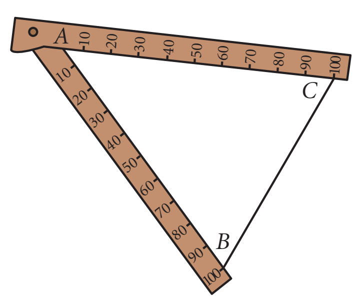


13. ***Construction*** Draw segment *EF*. Use compass and straightedge to divide it into five equal parts.

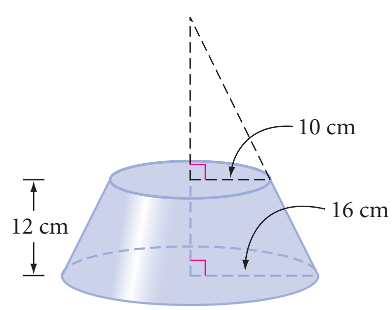
14. ***Construction*** Draw segment *IJ*. Construct a regular hexagon with *IJ* as the perimeter.

15. You can use a sheet of lined paper to divide a segment into equal parts. Draw a segment on a piece of patty paper, and divide it into five equal parts by placing it over lined paper. What conjecture explains why this works?

16. The drafting tool shown at right is called a sector compass. You position a given segment between the 100-marks. What points on the compass should you connect to construct a segment that is three-fourths (or 75%) of *BC*? Explain why this works.



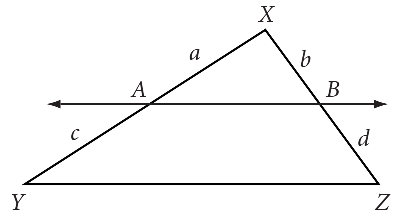
17. This truncated cone was formed by cutting off the top of a cone with a slice parallel to the base of the cone. What is the volume of the truncated cone? ⓗ



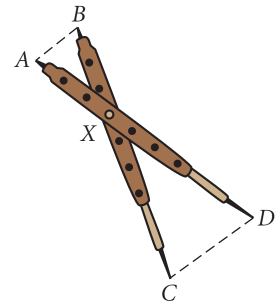
18. Assume that the SAS Similarity Conjecture and the Converse of the Parallel Lines Conjecture are true. Write a proof to show that if a line cuts two sides of a triangle proportionally, then it is parallel to the third side.

Given: $\frac{a}{c} = \frac{b}{d}$ (Assume $c \neq 0$ and $d \neq 0$.)

Show: $\overleftrightarrow{AB} \parallel \overline{YZ}$

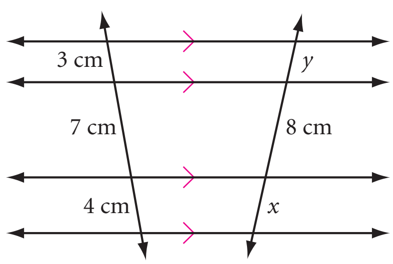


19. Another drafting tool used to construct segments is a pair of proportional dividers, shown at right. Two styluses of equal length are connected by a screw. The tool is adjusted for different proportions by moving the screw. Where should the screw be positioned so that AB is three-fourths of CD?

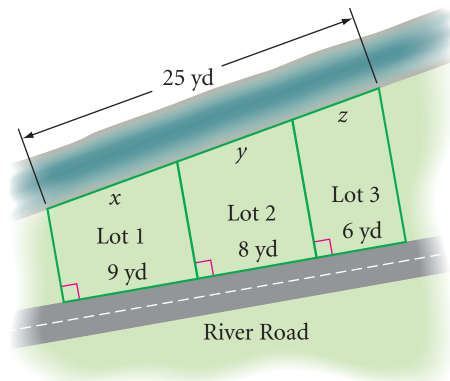


The Extended Parallel/Proportionality Conjecture can be extended even further. That is, you don't necessarily need a triangle. If three or more parallel lines intercept two other lines (transversals) in the same plane, they do so proportionally. For Exercises 20 and 21 use this extension.

20. Find x and y.



21. A real estate developer has parceled land between a river and River Road as shown. The land has been divided by segments perpendicular to the road. What is the "river frontage" (lengths x, y, and z) for each of the three lots?



Review

22. The ratio of the surface areas of two cubes is $\frac{49}{81}$. What is the ratio of their volumes?

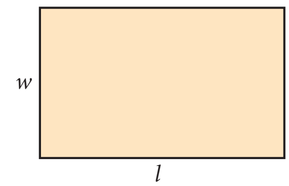
23. Romunda's original recipe for her special "cannonball" cookies makes 36 spheres with 4 cm diameters. She reasons that she can make 36 cannonballs with 8 cm diameters by doubling the amount of dough. Is she correct? If not, how many 8 cm diameter cannonballs can she make by doubling the recipe?

24. Find the surface area of a cube with edge x. Find the surface area of a cube with edge $2x$. Find the surface area of a cube with edge $3x$.

25. A circle of radius r has a chord of length r. Find the length of the minor arc.

26. ***Technology*** In Lesson 11.3, Exercise 17, you learned about the golden cut and the golden ratio. A **golden rectangle** is a rectangle in which the ratio of the length to the width is the golden ratio. That is, a golden rectangle's length, l, and width, w, satisfy the proportion

$$\frac{w}{l} = \frac{l}{w + l}$$

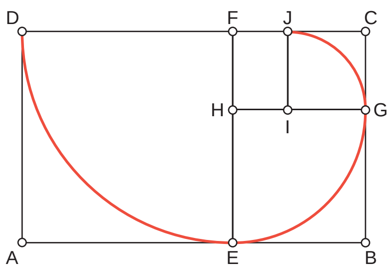


A golden rectangle

Some researchers believe Greek architects used golden rectangles to design the Parthenon. You can learn more about the historical importance of golden rectangles using the links at www.keymath.com/DG .

a. Use geometry software to construct a golden rectangle. Your construction for Exercise 17 in Lesson 11.3 will help.

b. When a square is cut off one end of a golden rectangle, the remaining rectangle is a smaller, similar golden rectangle. If you continue this process over and over again, and then connect opposite vertices of the squares with quarter-circles, you create a curve called the golden spiral. Use geometry software to construct a **golden spiral.** The first three quarter-circles are shown below.



ABCD is a golden rectangle.

EBCF is a golden rectangle.

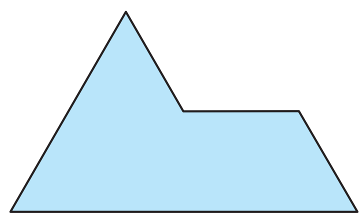
HGCF is a golden rectangle.

IJFH is a golden rectangle.

The curve from *D* to *E* to *G* to *J* is the beginning of a golden spiral.

27. A circle is inscribed in a quadrilateral. Write a proof showing that the two sums of the opposite sides of the quadrilateral are equal.

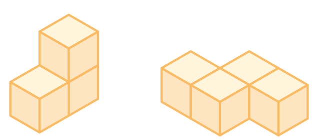
28. Copy the figure at right onto your own paper. Divide it into four figures similar to the original figure.



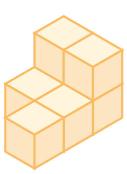
IMPROVING YOUR VISUAL THINKING SKILLS

Connecting Cubes

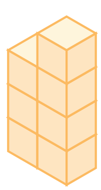
The two objects shown at right can be placed together to form each of the shapes below except one. Which one?



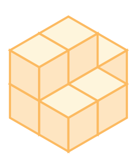
A.



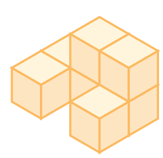
B.



C.



D.



Exploration

Two More Forms of Valid Reasoning

In the Chapter 10 Exploration Sherlock Holmes and Valid Forms of Reasoning, you learned about *Modus Ponens* and *Modus Tollens*. A third valid form of reasoning is called the Law of Syllogism.

According to the **Law of Syllogism** (LS), if you accept "If P then Q" as true and if you accept "If Q then R" as true, then you must logically accept "If P then R" as true.

Here is an example of the Law of Syllogism.

English statement	Symbolic translation
If I eat pizza after midnight, then I will have nightmares. If I have nightmares, then I will get very little sleep. Therefore, if I eat pizza after midnight, then I will get very little sleep.	P: I eat pizza after midnight. Q: I will have nightmares R: I will get very little sleep. $P \rightarrow Q$ $Q \rightarrow R$ $\therefore P \rightarrow R$

To work on the next law, you need some new statement forms. Every conditional statement has three other conditionals associated with it. To get the converse of a statement, you switch the "if" and "then" parts. To get the **inverse,** you negate both parts. To get the **contrapositive,** you reverse and negate the two parts. These new forms may be true or false.

Statement	If two angles are vertical angles, then they are congruent.	$P \rightarrow Q$	true
Converse	If two angles are congruent, then they are vertical angles.	$Q \rightarrow P$	false
Inverse	If two angles are not vertical angles, then they are not congruent.	$\sim P \rightarrow \sim Q$	false
Contrapositive	If two angles are not congruent, then they are not vertical angles.	$\sim Q \rightarrow \sim P$	true

Notice that the original conditional statement and its contrapositive have the same truth value. This leads to a fourth form of valid reasoning. The **Law of Contrapositive** (LC) says that if a conditional statement is true, then its contrapositive is also true. Conversely, if the contrapositive is true, then the original conditional statement must also be true. This also means that if a conditional statement is false, so is its contrapositive.

Often, a logical argument contains multiple steps, applying the same rule more than once, or applying more than one rule. Here is an example.

English statement	Symbolic translation
If the consecutive sides of a parallelogram are congruent, then it is a rhombus. If a parallelogram is a rhombus, then its diagonals are perpendicular bisectors of each other. The diagonals are not perpendicular bisectors of each other. Therefore the consecutive sides of the parallelogram are not congruent.	P: The consecutive sides of a parallelogram are congruent. Q: The parallelogram is a rhombus. R: The diagonals are perpendicular bisectors of each other. $P \to Q$ $Q \to R$ $\sim R$ $\therefore \sim P$

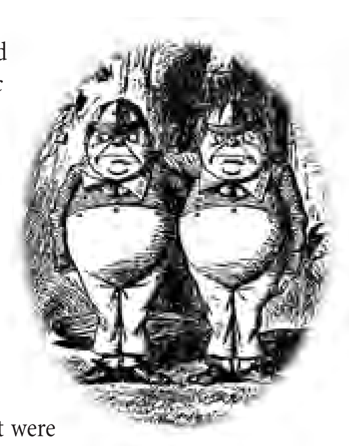
You can show that this argument is valid in three logical steps.

Step 1	$P \to Q$ $Q \to R$ $\therefore P \to R$	by the Law of Syllogism
Step 2	$P \to R$ $\therefore \sim R \to \sim P$	by the Law of Contrapositive
Step 3	$\sim R \to \sim P$ $\sim R$ $\therefore \sim P$	by *Modus Ponens*

Literature CONNECTION

Lewis Carroll was the pseudonym of the English novelist and mathematician Charles Lutwidge Dodgson (1832–1898). He is often associated with his famous children's book *Alice's Adventures in Wonderland.*

In 1886 he published *The Game of Logic,* which used a game board and counters to solve logic problems. In 1896 he published *Symbolic Logic, Part I,* which was an elementary book intended to teach symbolic logic. Here is one of the silly problems from *Symbolic Logic.* What conclusion follows from these premises?



> Babies are illogical.
> Nobody is despised who can manage a crocodile.
> Illogical persons are despised.

Lewis Carroll enjoyed incorporating mathematics and logic into all of his books. Here is a quote from *Through the Looking Glass.* Is Tweedledee using valid reasoning?

> "Contrariwise," said Tweedledee, "if it was so, it might be; and if it were so, it would be, but as it isn't, it ain't. That's logic."

So far, you have learned four basic forms of valid reasoning.

Now let's apply them in symbolic proofs.

Four Forms of Valid Reasoning

$P \to Q$	$P \to Q$	$P \to Q$	$P \to Q$
P	$\sim Q$	$Q \to R$	$\therefore \sim Q \to \sim P$
$\therefore Q$	$\therefore \sim P$	$\therefore P \to R$	
by MP	by MT	by LS	by LC

Activity

Symbolic Proofs

Step 1 Determine whether or not each logical argument is valid. If it is valid, state what reasoning form or forms it follows. If it is not valid, write "no valid conclusion."

a. $P \to \sim Q$
Q
$\therefore \sim P$

b. $\sim S \to P$
$R \to \sim S$
$\therefore R \to P$

c. $\sim Q \to \sim R$
$\sim Q$
$\therefore \sim R$

d. $R \to P$
$T \to \sim P$
$\therefore R \to T$

e. $\sim P \to \sim R$
R
$\therefore P$

f. $P \to Q$
$\sim R \to \sim Q$
$\therefore P \to R$

Step 2 Translate parts a–c into symbols, and give the reasoning form(s) or state that the conclusion is not valid.

a. If I study all night, then I will miss my late-night talk show. If Jeannine comes over to study, then I study all night. Jeannine comes over to study. Therefore I will miss my late-night talk show.

b. If I don't earn money, then I can't buy a computer. If I don't get a job, then I don't earn money. I have a job. Therefore I can buy a computer.

c. If $\overline{EF}$ is not parallel to side $\overline{AB}$ in trapezoid *ABCD*, then $\overline{EF}$ is not a midsegment of trapezoid *ABCD*. If $\overline{EF}$ is parallel to side $\overline{AB}$, then *ABFE* is a trapezoid. $\overline{EF}$ is a midsegment of trapezoid *ABCD*. Therefore *ABFE* is a trapezoid.

Step 3 Show how you can use *Modus Ponens* and the Law of Contrapositive to make the same logical conclusions as *Modus Tollens*.

CHAPTER 11 REVIEW

Similarity, like area, volume, and the Pythagorean Theorem, has many applications. Any scale drawing or model, anything that is reduced or enlarged, is governed by the properties of similar figures. So engineers, visual artists, and film-makers all use similarity. It is also useful in indirect measurement. Do you recall the two indirect measurement methods you learned in this chapter? The ratios of area and volume in similar figures are also related to the ratios of their dimensions. But recall that as the dimensions increase, the area increases by a squared factor and volume increases by a cubed factor.

Exercises

You will need

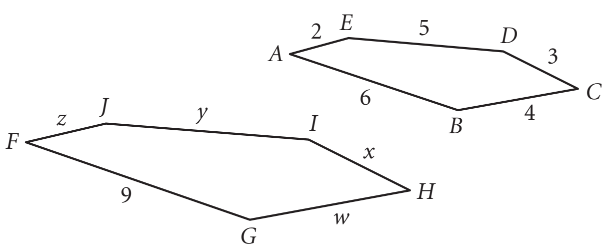
Construction tools for Exercise **9**

For Exercises 1–4, solve each proportion.

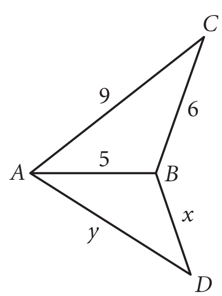
1. $\frac{x}{15} = \frac{8}{5}$ **2.** $\frac{4}{11} = \frac{24}{x}$ **3.** $\frac{4}{x} = \frac{x}{9}$ **4.** $\frac{x}{x+3} = \frac{34}{40}$

In Exercises 5 and 6, measurements are in centimeters.

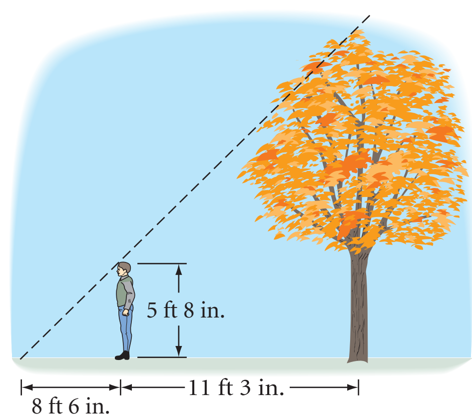
5. $ABCDE \sim FGHIJ$
$w = \underline{\ ?\ }$, $x = \underline{\ ?\ }$, $y = \underline{\ ?\ }$, $z = \underline{\ ?\ }$



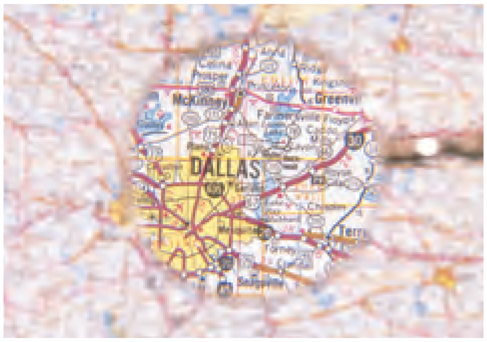
6. $\triangle ABC \sim \triangle DBA$
$x = \underline{\ ?\ }$, $y = \underline{\ ?\ }$



7. **APPLICATION** David is 5 ft 8 in. tall and wants to find the height of an oak tree in his front yard. He walks along the shadow of the tree until his head is in a position where the end of his shadow exactly overlaps the end of the tree's shadow. He is now 11 ft 3 in. from the foot of the tree and 8 ft 6 in. from the end of the shadows. How tall is the oak tree?

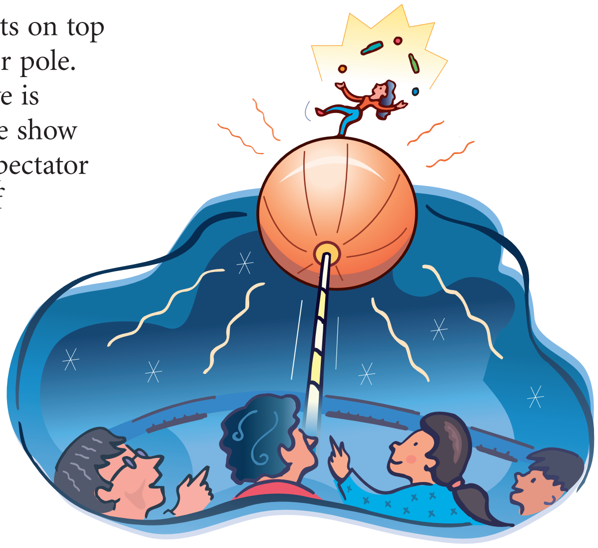


8. A certain magnifying glass when held 6 in. from an object creates an image that is 10 times the size of the object being viewed. What is the measure of a 20° angle under this magnifying glass?



9. ***Construction*** Construct $\overline{KL}$. Then find a point P that divides $\overline{KL}$ into two segments that have a ratio $\frac{3}{4}$. ⓗ

10. Patsy does a juggling act. She sits on a stool that sits on top of a rotating ball that spins at the top of a 20-meter pole. The diameter of the ball is 4 meters, and Patsy's eye is approximately 2 meters above the ball. Seats for the show are arranged on the floor in a circle so that each spectator can see Patsy's eyes. Find the radius of the circle of seats to the nearest meter. ⓗ



11. Charlie builds a rectangular box home for his pet python and uses 1 gallon of paint to cover its surface. Lucy also builds a box for Charlie's pet, but with dimensions twice as great. How many gallons of paint will Lucy need to paint her box? How many times as much volume does her box have?

12. Suppose you had a real clothespin similar to the sculpture at right and made of the same material. What measurements would you make to calculate the weight of the sculpture? Explain your reasoning.

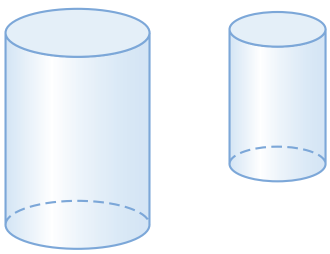
13. The ratio of the perimeters of two similar parallelograms is $\frac{3}{7}$. What is the ratio of their areas?

14. The ratio of the areas of two circles is $\frac{25}{16}$. What is the ratio of their radii?

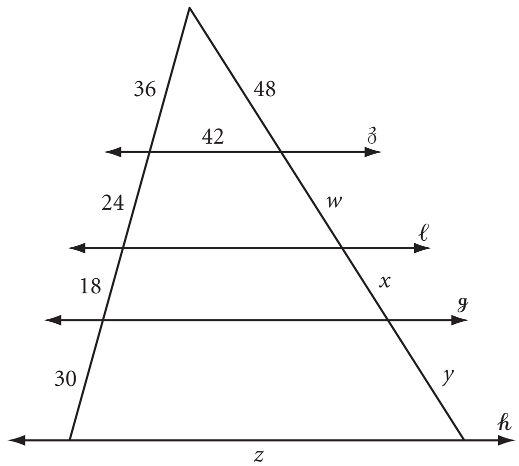
15. **APPLICATION** The Jones family paid \$150 to a painting contractor to stain their 12-by-15-foot deck. The Smiths have a similar deck that measures 16 ft by 20 ft. What price should the Smith family expect to pay to have their deck stained?

This sculpture, called *Clothespin* (1976), was created by Swedish-American sculptor Claes Oldenburg (b 1929). His art reflects how everyday objects can be intriguing.

16. The dimensions of the smaller cylinder are two-thirds of the dimensions of the larger cylinder. The volume of the larger cylinder is 2160π cm^3. Find the volume of the smaller cylinder.



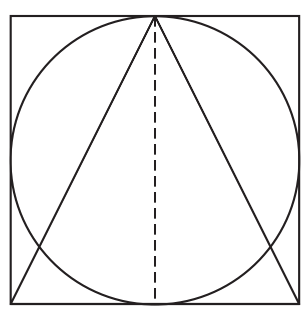
17. $\mathfrak{z} \parallel \ell \parallel \mathfrak{g} \parallel \mathfrak{h}$
$w = \underline{\ ?\ }$, $x = \underline{\ ?\ }$, $y = \underline{\ ?\ }$, $z = \underline{\ ?\ }$



18. Below is a 58-foot statue of Bahubali, in Sravanabelagola, India. Every 12 years, worshipers of the Jain religion bathe the statue with coconut milk. Suppose the milk of one coconut is just enough to cover the surface of the similar 2-foot statuette shown at right. How many coconuts would be required to cover the surface of the full-size statue?

This 58-foot statue is carved from a single stone.

19. Greek mathematician Archimedes liked the design at right so much that he wanted it on his tombstone. ⓗ

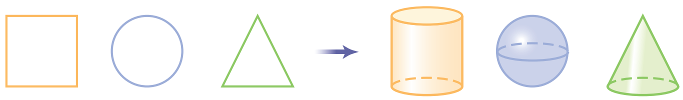


a. Calculate the ratio of the area of the square, the area of the circle, and the area of the isosceles triangle. Copy and complete this statement of proportionality.

Area of square to Area of circle to Area of triangle is $\underline{\ ?\ }$ to $\underline{\ ?\ }$ to $\underline{\ ?\ }$.

b. When each of the figures is revolved about the vertical line of symmetry, it generates a solid of revolution—a cylinder, a sphere, and a cone. Calculate their volumes. Copy and complete this statement of proportionality.

Volume of cylinder to Volume of sphere to Volume of cone is _?_ to _?_ to _?_.



c. What is so special about this design?

20. Many fanciful stories are about people who accidentally shrink to a fraction of their original height. If a person shrank to one-twentieth his original height, how would that change the amount of food he'd require, or the amount of material needed to clothe him, or the time he'd need to get to different places? Explain.

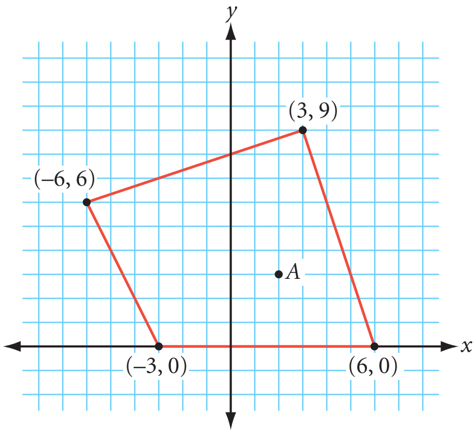
This scene is from the 1957 science fiction movie *The Incredible Shrinking Man.*

21. Would 15 pounds of 1-inch ice cubes melt faster than a 15-pound block of ice? Explain.

TAKE ANOTHER LOOK

1. You've learned that an ordered pair rule such as $(x, y) \rightarrow (x + b, y + c)$ is a translation. You discovered in this chapter that an ordered pair rule such as $(x, y) \rightarrow (kx, ky)$ is a dilation in the coordinate plane, centered at the origin. What transformation is described by the rule $(x, y) \rightarrow (kx + b, ky + c)$? Investigate.

2. In Lesson 11.1, you dilated figures in the coordinate plane, using the origin as the center of dilation. What happens if a different point in the plane is the center of dilation? Copy the polygon at right onto graph paper. Draw the polygon's image under a dilation with a scale factor of 2 and with point A as the center of dilation. Draw another image using a scale factor of $\frac{2}{3}$. Explain how you found the image points. How does dilating about point A differ from dilating about the origin?

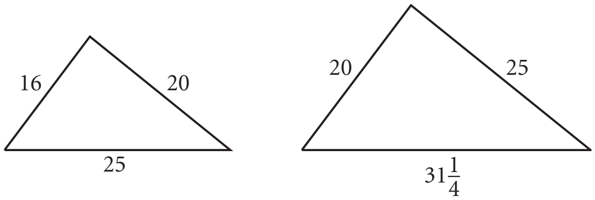


3. True or false? The angle bisector of one of the nonvertex angles of a kite will divide the diagonal connecting the vertex angles into two segments whose lengths are in the same ratio as two unequal sides of the kite. If true, explain why. If false, show a counterexample that proves it false.

4. A total eclipse of the Sun can occur because the ratio of the Moon's diameter to its distance from Earth is about the same as the ratio of the Sun's diameter to its distance to Earth. Draw a diagram and use similar triangles to explain why it works.

5. It is possible for the three angles and two of the sides of one triangle to be congruent to the three angles and two of the sides of another triangle, and yet the two triangles won't be congruent. Two such triangles are shown below. Use geometry software or patty paper to find another pair of similar (but not congruent) triangles in which five parts of one are congruent to five parts of the other.


A solar eclipse



Explain why these sets of side lengths work. Use algebra to explain your reasoning.

6. Is the converse of the Extended Parallel Proportionality Conjecture true? That is, if two lines intersect two sides of a triangle, dividing the two sides proportionally, must the two lines be parallel to the third side? Prove that it is true or find a counterexample showing that it is not true.

7. If the three sides of one triangle are each parallel to one of the three sides of another triangle, what might be true about the two triangles? Use geometry software to investigate. Make a conjecture and explain why you think your conjecture is true.

Assessing What You've Learned

UPDATE YOUR PORTFOLIO If you did the Project Making a Mural, add your mural to your portfolio.

ORGANIZE YOUR NOTEBOOK Review your notebook to be sure it's complete and well organized. Be sure you have each definition and the conjecture. Write a one-page summary of Chapter 11.

GIVE A PRESENTATION Give a presentation about one or more of the similarity conjectures. You could even explain how an overhead projector produces similar figures!